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Vol. 12 | Mathematical Olympiad Series

Geometric Inequalities

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Contents

Chapter 1	The method of segment replacement for distance inequalities	1
Chapter 2	Ptolemy's inequality and its application	12
Chapter 3	Inequality for the inscribed quadrilateral	22
Chapter 4	The area inequality for special polygons	36
Chapter 5	Linear geometric inequalities	52
Chapter 6	Algebraic methods	64
Chapter 7	Isoperimetric and extremal value problem	73
Chapter 8	Embed inequality and inequality for moment of inertia	81
Chapter 9	Locus problem of Tsintsifas's inequality	95
Chapter 10	Shum's minimal circle problem	100
Chapter 11	Inequalities for tetrahedron	108
Answers and hints to selected exercises		119

Preface

"God is always doing geometry", said Plato. But the deep investigation and extensive attention to geometric inequalities as an independent field is a matter of modern times.

Many geometric inequalities are not only typical examples of mathematical beauty but also tools for application as well. The well known Brunn-Minkowski's inequality is such an example. "It is like a large octopus, whose tentacles stretches out into almost every field of mathematics. It has not only relation with advanced mathematics such as the Hodge index theorem in algebraic geometry, but also plays an important role in applied subjects such as stereology, statistical mechanics and information theory".

There are dozens of books on geometric inequalities so far, in which "Geometric Inequalities" by Yu. D. Burago and V. A. Zalgaller is cited worldwide. And "Geometric Inequalities" by Chinese scholar Mr. San Zhun is an excellent introductory book (Shanghai Education Press, 1980).

The aim of this book is mainly to introduce geometric inequalities to students and high school teachers who wish to attend the Mathematics Olympiad Competition. The material is elementary. In the process of writing, I strive to achieve: firstly, carefully select new achievements, methods and techniques of recent studies. Secondly, the material should be simple but non-trivial, with an interesting and profound background. Thirdly, as far as possible to present the students' excellent answers and, of course, also some results on my research and experiences. Any comments and suggestions are welcome.

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I dedicate this book to Mr. Qiu Zonghu as a form of congratulations on his seventieth birthday and also to commemorate his great contributions to the Mathematical Olympiad of China.

At last, I would like to thank Mr. Ni Ming for his faithfulness and patience in the publication of this book. And thanks also give to my doctoral student Mr. Si Lin for his typist and drawings.

My cherished desire is that the readers like this book.

Leng Gangsong
October 2004

Chapter 1 The method of segment replacement for distance inequalities

The comparison of lengths is more basic than comparison of other geometric quantities (such as angles, areas and volumes). A geometric inequality that involves only the lengths is called a distance inequality.

Some simple axioms and theorems on inequalities in Euclidean geometry are usually the starting point to solve problems of distance inequality, in which most frequently used tools are:

Proposition 1. The shortest line connecting point A with point B is the segment AB.

The direct corollary of Proposition 1 is

Proposition 2 (Triangle Inequality). For arbitrary three points A, B and C, we have $AB \leqslant AC + CB$, the equality holds if and only if C is on the segment AB.

Remark. In this book, to simplify notations, any symbol of geometric object also denotes its quantity according to the context.

Proposition 2 has the following often used consequences.

Proposition 3. In a triangle, the longer side has the larger opposite angle. And the larger angle has the longer opposite side.

Proposition 4. The median of a triangle on a side is shorter than the half-sum of the other two sides.

Proposition 5. If a convex polygon is within another one, then the outside convex polygon's perimeter is larger.

Proposition 6. Any segment in a convex polygon is either less than the longest side or the longest diagonal of the convex polygon.

Firstly, we give an example.

Example 1. Let a, b and c be sides of $\triangle ABC$. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2. \tag{1}$$

Proof. By the triangle inequality $a<b+c$, yields

$$\frac{a}{b+c}=\frac{2a}{2(b+c)}<\frac{2a}{a+b+c}.$$

Similarly,

$$\frac{b}{c+a}<\frac{2b}{a+b+c},$$
$$\frac{c}{b+a}<\frac{2c}{a+b+c}.$$

Adding up the above three inequalities leads to Inequality (1). □

Example 2. Let AB be the longest side of $\triangle ABC$, and P a point in the triangle, prove that

$$PA+PB>PC. \tag{2}$$

Proof. Let D be the intersection point of CP and AB (see Figure 1.1), then $\angle ADC$ or $\angle BDC$ is not acute. Without loss of generality, we assume that $\angle ADC$ is not acute. Applying Proposition 3 to $\triangle ADC$, we obtain

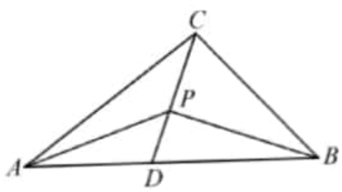

Figure 1.1

$$AC \geqslant CD.$$

Therefore,

$$AB \geqslant AC \geqslant CD > PC. \tag{a}$$

triangle inequality to $\triangle PAB$, we have

$$AC \geqslant CD.$$

Therefore,

$$AB \geqslant AC \geqslant CD > PC. \tag{a}$$

Furthermore, applying triangle inequality to $\triangle PAB$, we have

$$PA + PB > AB. \tag{b}$$

Combining (a) and (b), we obtain Inequality (2) immediately. □

Remark. (1) If AB is not the longest, then the conclusion may not be true.

(2) If point P on the plane of regular $\triangle ABC$, and P is not on the circumcircle of the triangle, then the sum of any two of PA, PB and PC is longer than the remaining one. That is, PA, PB and PC consist of a triangle's three sides.

Example 3. A closed polygonal line with perimeter 1 can be put inside a circle with radius $\frac{1}{4}$.

Analysis. The key to prove Example 3 is to determine the center O of the circle, such that the distance of point on the polygonal line to point O is less or equal to $\frac{1}{4}$.

Proof. Let points A and B be two arbitrary points that bisect the perimeter of the closed polygonal line (see Figure 1.2). That is, the length of the polygonal line $\widehat{AB}$ is $\frac{1}{2}$. Let the center of circle O be the midpoint of the segment AB, then the distance from each point on the closed polygonal line to point O is less than $\frac{1}{4}$.

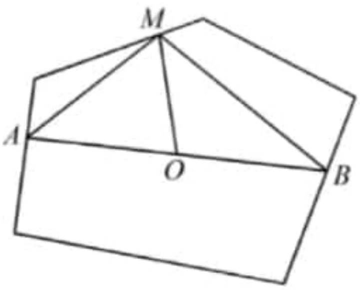

Figure 1.2

In fact, let M be a point on the closed polygonal line but not A or

B, applying Proposition 4, we have

$$OM < \frac{1}{2}(AM + MB) \leqslant \frac{1}{2}(\widehat{AM} + \widehat{BM}) < \frac{1}{2}\widehat{AB} = \frac{1}{4},$$

where symbols such as $\widehat{AM}$ denote the polygonal line with endpoints A and B, and its length as well.

And if M is A or B, then $OM = \frac{AB}{2} < \frac{1}{4}$ by Proposition 1.

Now, we draw a circle with center O and radius $\frac{1}{4}$, then the polygon is located inside the circle. □

In fact, the proofs of above examples embody an idea of "segment replacement", we call it "the method of segment replacement". Specifically, this method is based on Proposition 1 or its inference, replace curve to polygonal line, then replace polygonal line to segment. This method is one of the most commonly used methods for proving geometric inequalities, especially distance inequalities.

Now, here are other examples.

Firstly, we introduce the classical Pólya's problem.

Example 4. Of all the curves that can bisect the area of a circle and with their endpoints on its circumference, the diameter of the circle has the shortest length.

Proof. Denote the curve by $\widehat{AB}$, points A, B on the circle. If A and B are two endpoints of a diameter, then it is clear that $\widehat{AB}$ is not less than the diameter.

If chord AB is not a diameter (see Figure 1.3), let diameter CD be parallel to AB (if $A = B$, then CD can be any diameter that does not intersect with A or B), then curve $\widehat{AB}$ intersects CD at point E which is not the center, hence

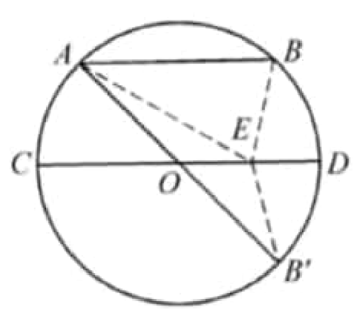

Figure 1. 3

$$\widehat{AB} = \widehat{AE} + \widehat{EB} \geqslant AE + EB.$$

The key point is that we consider the polygonal line instead of the curve.

Now we show that $\widehat{AE} + \widehat{EB} >$ the diameter of the circle. Let B' be the symmetric point of B to CD, then AB' is the diameter of the circle. So

$$AE + EB = AE + EB' > AB' = \text{the diameter of the circle.}$$

□

The following example stems from our research on extremal property of the pedal triangle.

Example 5. Let P be a point in $\triangle ABC$, and A', B' and C' be the projection of P onto BC, CA and AB or their extended, respectively; Let A'', B'' and C'' be the intersection points of AP, BP and CP to corresponding sides, respectively. And the perimeter of $\triangle A''B''C''$ equals 1. Prove that

$$\widehat{A'B''C'A''} + \widehat{A'C''B'A''} \leqslant 2.$$

Proof. The required inequality is equivalent to

$$A'B'' + B''C' + C'A'' + A'C'' + C''B' + B'A'' \leqslant 2. \tag{a}$$

To prove (a), we need only to prove the local inequality

$$A'B'' + A'C'' \leqslant A''B'' + A''C''. \tag{b}$$

Similarly,

$$B'A'' + B'C'' \leqslant B''A'' + B''C'',$$
$$C'A'' + C'B'' \leqslant C''A'' + C''B''.$$

Adding up these inequalities, we get (a) immediately. □

Before proceeding to prove (b) we need the following lemma.

Lemma 1. Let point P be on the altitude AD of $\triangle ABC$ (see Figure

1.4). If BP intersects AC at E, and CP intersects AB at F, then

$$\angle FDA = \angle EDA.$$

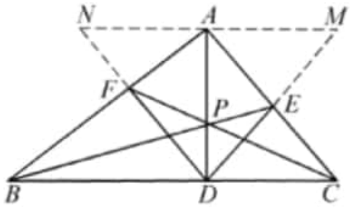

Figure 1.4

Proof. Suppose that the line parallel to BC intersects lines DE and DF at points M and N, respectively, then

$$\frac{AF}{BF} = \frac{AN}{BD}, \frac{CE}{AE} = \frac{CD}{AM}.$$

By Ceva's theorem,

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

that is,

$$AM = AN.$$

By $AD \perp MN$, we have $DM = DN$, so

$$\angle EDA = \angle ADM = \angle ADN = \angle FDA.$$

□

Now we turn to prove (b):

Proof. (i) If P is a point on the altitude AD of $\triangle ABC$, then $A' = A''$, obviously, (b) is true.

(ii) If P is not a point on the altitude AD of $\triangle ABC$ (see Figure 1.5), without loss of generality, we may assume that P, B lie on ipsilateral of AB, line $A'P$ intersects AB at M, MC intersects BB'' at M''. By Lemma 1 we have

$$\angle B''A'P > \angle M'A'P = \angle C''A'P. \quad \text{(c)}$$

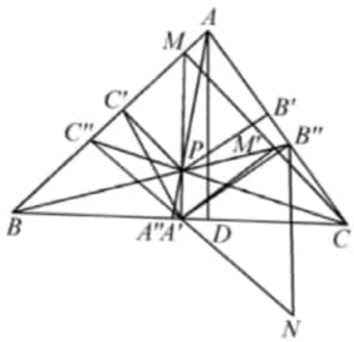

Figure 1.5

Let N be the symmetric point of B'' to BC, then

$$\angle NA'C = \angle CA'B''.$$

By (c) we have

$$\begin{aligned} & \angle NA'C + \angle C''A'C \\ = & \angle NA'C + \angle C''A'P + \angle PA'C \end{aligned}$$

M''. By Lemma 1 we have

$$\angle B''A'P > \angle M'A'P = \angle C''A'P. \quad (c)$$

Let N be the symmetric point of B'' to BC, then

Figure 1.5

$$\angle NA'C = \angle CA'B''.$$

By (c) we have

$$\begin{aligned} & \angle NA'C + \angle C''A'C \\ = & \angle NA'C + \angle C''A'P + \angle PA'C \\ < & \angle PA'B'' + \angle PA'N \\ = & \pi, \end{aligned}$$

so A' and A'' lie on ipsilateral of $C''N$, that is, A' is a point in $\triangle C''A''N$, then by Proposition 5 we have

$$A''C'' + A''N > A'C'' + A'N.$$

Notice that $A''B'' = A''N$, $A'B'' = A'N$, we have

$$A''B'' + A''C'' \geqslant A'B'' + A'C''.$$

Thus, we have proved (b). $\square$

Remark. (1) Symmetric reflection method in Example 5 is an often used means of segment replacement.

(2) Applying inequality (b), Dr. Yuan Jun proved a conjecture of Mr. Liu Jian:

The perimeter of $\triangle A'B'C' \leqslant$ the perimeter of $\triangle A''B''C''$.

The following example is a rather hard problem.

Example 6. Let P be a point in $\triangle ABC$, show that

$$\sqrt{PA} + \sqrt{PB} + \sqrt{PC} < \frac{\sqrt{5}}{2}(\sqrt{BC} + \sqrt{CA} + \sqrt{AB}). \quad \text{(a)}$$

First we state the following lemma which can be derived by Proposition 5 directly.

Lemma 2. Let P be a point in the convex quadrilateral $ABCD$, then

$$PB + PC < BA + AD + DC.$$

Next we prove (a).

Proof. Let $BC = a$, $AC = b$, $BA = c$, $PA = x$, $PB = y$ and $PC = z$ (see Figure 1.6), and let A', B' and C' be midpoints of the sides of $\triangle ABC$, then P must be in one of the parallelograms $A'B'AC'$, $C'B'CA'$ and $B'A'BC'$. Without loss of generality, we can assume that P is in parallelogram $A'B'AC'$, then applying Lemma 2 to convex quadrilateral $ABA'B'$, we have

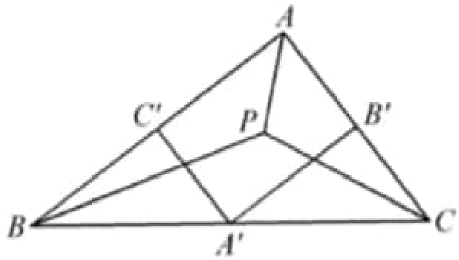

Figure 1.6

$$PA + PB < BA' + A'B' + B'A,$$

that is

$$x + y < \frac{1}{2}(a + b + c). \tag{b}$$

Similarly, for convex quadrilateral $ACA'C'$, we have

$$PA + PC < AC' + C'A' + A'C,$$

that is

$$x + z < \frac{1}{2}(a + b + c). \tag{c}$$

Adding up (b) and (c), we find that

$$2x + y + z < a + b + c. \tag{d}$$

Now we notice that the original inequality is equivalent to

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 < \frac{5}{4}(\sqrt{a} + \sqrt{b} + \sqrt{c})^2,$$

that is

$$\begin{aligned} &x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz} \\ &< \frac{5}{4}(a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}). \end{aligned} \tag{e}$$

Thus, it suffices to prove (e).

By the mean value inequality, we have

$$2\sqrt{xy} \leqslant 2x + \frac{1}{2}y,$$

$$2\sqrt{xz} \leqslant 2x + \frac{1}{2}z,$$

$$2\sqrt{yz} \leqslant y + z.$$

Combining these three inequalities and inequality (d), we get

$$\begin{aligned}\text{the left side of (e)} &\leqslant x + y + z + 2x + \frac{1}{2}y + 2x + \frac{1}{2}z + y + z \\ &= \frac{5}{2}(2x + y + z) < \frac{5}{2}(a + b + c),\end{aligned}$$

so we need only to prove

$$\frac{5}{2}(a + b + c) < \frac{5}{4}(a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}). \quad \text{(f)}$$

But (f) is equivalent to

$$a + b + c < 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ac}), \quad \text{(g)}$$

which is a simple inequality. In fact, without loss of generality, suppose that $a \geqslant b \geqslant c$, then by $b + c > a$,

$$\text{the right side of (g)} > 2(b + c) > a + b + c = \text{the left side of (g)}.$$

Therefore, (a) has been proved. □

Remark. (1) The constant $\sqrt{5}/2$ of inequality (a) is optimal, the proof of which is left to the reader.

(2) The elegant answer above was given by Zhu Qingsan (the former student of High School Affiliated to South China Normal University, who won a gold medal at the 45th IMO in 2004). Smartly positioning point P and dealing well with the non-fully symmetry variable are the key points to the answer.

Of course, Example 6 can also be proved by contour line. Contour

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Of course, Example 6 can also be proved by contour line. Contour

line is a special plane curve, such as circle, ellipse and so on, introduced to discuss extremal problems. Here we use ellipse as the contour line.

An Alternative Proof of Example 6

Proof. Let $BC = a$, $CA = b$ and $AB = c$. Without loss of generality, suppose that $a \leqslant b$ and $a \leqslant c$.

Now, we make an ellipse through P with focal points B and C, and intersects AB and AC at E and F, respectively (see Figure 1.7), then by Proposition 1 we have

$$PA \leqslant \max(EA, FA).$$

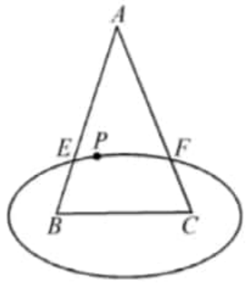

Figure 1.7

Without loss of generality, suppose that $EA \geqslant FA$, then $PA \leqslant EA$. Further,

$$\sqrt{PC} + \sqrt{PB} \leqslant \sqrt{2(PB + PC)} = \sqrt{2(EB + EC)},$$

therefore

$$\begin{aligned} & \sqrt{PA} + \sqrt{PB} + \sqrt{PC} \\ < {} & \sqrt{EA} + \sqrt{2(EB + EC)} \\ \leqslant {} & \left[5EA + \frac{5}{2}(EB + EC)\right]^{\frac{1}{2}} \\ = {} & \left[5(EA + EB) + \frac{5}{2}(EC - EB)\right]^{\frac{1}{2}} \\ < {} & \sqrt{5}\sqrt{c + \frac{a}{2}} \\ < {} & \frac{\sqrt{5}}{2}(\sqrt{a} + \sqrt{b} + \sqrt{c}). \end{aligned}$$

□

Chapter 2 Ptolemy's inequality and its application

The famous Ptolemy's inequality is a distance inequality for arbitrary quadrilateral. It can be written as

Theorem 1 (Ptolemy's inequality). In the quadrilateral $ABCD$, we have

$$AB \cdot CD + AD \cdot BC \geqslant AC \cdot BD,$$

the equality holds if and only if four points A, B, C and D lie on a circle.

Proof. Let E be a point in quadrilateral $ABCD$ (see Figure 2.1), such that $\angle BAE = \angle CAD$, $\angle ABE = \angle ACD$, then $\triangle ABE \backsim \triangle ACD$. So $AB \cdot CD = AC \cdot BE$. Also $\angle BAC = \angle EAD$, and $AB/AE = AC/AD$, then $\triangle ABC \backsim \triangle AED$, $AD \cdot BC = AC \cdot DE$. So

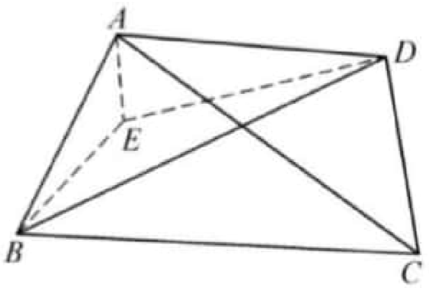

Figure 2.1

$$AB \cdot CD + AD \cdot BC = AC(BE + DE) \geqslant AC \cdot BD,$$

the equality holds if and only if point E is on segment BD. Thus $\angle ABD = \angle ACD$, so $ABCD$ is an inscribed quadrilateral. □

By applying Ptolemy's inequality, we have simple proofs for some distance inequalities.

Example 1 (Klamkin's dual inequality). Let a, b and c be the three sides of $\triangle ABC$, and let m_b and m_c be medians of b and c, respectively.

Prove that

$$4m_a m_b \leqslant 2a^2 + bc. \tag{1}$$

Proof. Construct parallelogram $ABCD$ and $ACBE$ (see Figure 2.2), connect BD and CE. Notice that $DE = 2a$, $BD = 2m_a$, and $CE = 2m_c$, applying Ptolemy's inequality for quadrilateral $BCDE$,

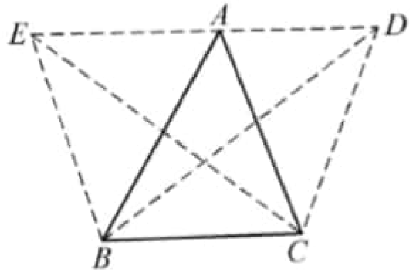

Figure 2.2

$$BC \cdot DE + BE \cdot CD \geqslant BD \cdot EC.$$

That is (1). □

Example 2. Let a, b and c be the three sides of $\triangle ABC$, and let m_a, m_b, and m_c be medians on sides a, b and c, respectively. Prove that

$$m_a(bc - a^2) + m_b(ac - b^2) + m_c(ab - c^2) \geqslant 0. \tag{2}$$

The key to the following proof is to find a special quadrilateral.

Proof. Let AD, BE and CF be medians of triangle ABC with barycenter G (see Figure 2.3).

Applying Ptolemy's inequality to quadrilateral $BDGF$,

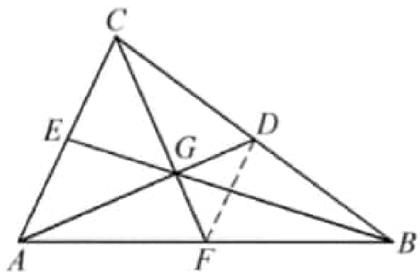

Figure 2.3

$$BG \cdot DF \leqslant GF \cdot DB + DG \cdot BF. \text{ (a)}$$

Notice that $BG = \dfrac{2}{3}m_b$, $DG = \dfrac{1}{3}m_a$, $GF = \dfrac{1}{3}m_c$ and $DF = \dfrac{b}{2}$, (a) can be rewritten into:

$$2bm_b \leqslant am_c + cm_a.$$

So

$$2b^2 m_b \leqslant abm_c + cbm_a. \tag{b}$$

Similarly,

$$2c^2 m_c \leqslant acm_b + bcm_a, \tag{c}$$

$$2a^2m_a \leqslant abm_c + acm_b. \tag{d}$$

Adding up (b), (c) and (d), we have

$$2(m_a bc + m_b ca + m_c ab) \geqslant 2(m_a a^2 + m_b b^2 + m_c c^2),$$

and by rearranging terms, we obtain inequality (2). □

Like Example 2, the following is another example of geometric linear inequality generated by Ptolemy's theorem.

Example 3. Let $A_1A_2\cdots A_n$ be a regular n-polygon, and M_1, M_2, ..., M_n be midpoints of the corresponding sides. Let P be an arbitrary point in the plane which n-polygon lies in.

Prove that

$$\sum_{i=1}^{n} PM_i \geqslant \left(\cos\frac{\pi}{n}\right)\sum_{i=1}^{n} PA_i. \tag{3}$$

Proof. Let M_{i-1}, M_i be midpoints of the $(i-1)$th and (i)th edge of the regular n-polygon, respectively (see Figure 2.4). Applying Ptolemy's inequality to quadrilateral $PM_{i-1}A_iM_i$, we obtain the partial inequality

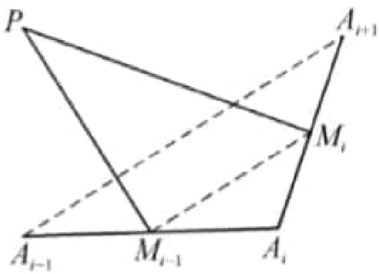

Figure 2.4

$$A_iM_{i-1} \cdot PM_i + PM_{i-1} \cdot A_iM_i \geqslant PA_i \cdot M_{i-1}M_i,$$

so

$$PM_i + PM_{i-1} \geqslant 2\left(\cos\frac{\pi}{n}\right) \cdot PA_i, \tag{a}$$

where $i = 1, 2, \ldots, n$ and $A_0 = A_n$, $M_0 = M_n$.

Now summation on both sides of (a), we have

$$\sum_{i=1}^{n}(PM_i + PM_{i-1}) \geqslant 2\left(\cos\frac{\pi}{n}\right) \cdot \sum_{i=1}^{n} PA_i,$$

which is equivalent to inequality (3). □

The following two examples introduce skills of dealing with

$$AG + GB + GH + DH + HE \geqslant CF.$$

(36th IMO problem)

4. Let $\triangle ABC$ be inscribed on $\odot O$, and P be an arbitrary point in $\triangle ABC$. Construct parallel lines of AB, AC, BC through P, intersect BC, AC at F, E, intersect AB, BC at K, I, intersect AB, AC at G, H respectively. Let AD be a chord of $\odot O$ through P, prove that

$$EF^2 + KI^2 + GH^2 \geqslant 4PA \cdot PD.$$

5. In $\triangle ABC$, let bisectors of $\angle A$, $\angle B$, $\angle C$ intersect circumcircle of $\triangle ABC$ at A_1, B_1 and C_1, respectively. Prove that

$$AA_1 + BB_1 + CC_1 > AB + BC + CA.$$

(Australian Competition in 1982)

Chapter 3 Inequality for the inscribed quadrilateral

Inscribed quadrilateral has not only rich relationships in geometric equalities, but also possesses interesting extremal properties. Since the sides of the inscribed quadrilateral can be represented by the trigonometric functions of the corresponding central angle, this makes it possible to use the trigonometric method to do with the geometric inequalities of the inscribed quadrilateral. The following is such an example.

Example 1. Let $ABCD$ be an inscribed quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geqslant 2|AC - BD|. \tag{1}$$

(Problem of The 28th Mathematical Olympiad of America)

Proof. Let O be the circumcenter of the inscribed quadrilateral $ABCD$ with radius 1, (see Figure 3.1) $\angle AOB = 2\alpha$, $\angle BOC = 2\beta$, $\angle COD = 2\gamma$, $\angle DOA = 2\delta$, then

$$\alpha + \beta + \gamma + \delta = \pi.$$

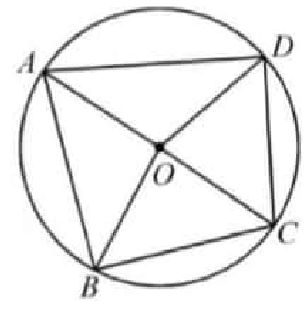

Figure 3.1

Without loss of generality assume that $\alpha \geqslant \gamma$, $\beta \geqslant \delta$, it follows that

$$\begin{aligned} |AB - CD| &= 2|\sin\alpha - \sin\gamma| \\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\cos\frac{\alpha+\gamma}{2}\right| \\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta+\delta}{2}\right|. \end{aligned}$$

Similarly, we have

$$|AD - BC| = 4\left|\sin\frac{\beta-\delta}{2}\sin\frac{\alpha+\gamma}{2}\right|,$$

$$|AC - BD| = 4\left|\sin\frac{\beta-\delta}{2}\sin\frac{\alpha-\gamma}{2}\right|.$$

Therefore

$$\begin{aligned}|AB - CD| - |AC - BD| &= 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left(\left|\sin\frac{\beta+\delta}{2}\right| - \left|\sin\frac{\beta-\delta}{2}\right|\right)\\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left(\sin\frac{\beta+\delta}{2} - \sin\frac{\beta-\delta}{2}\right)\\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\right|\cdot\left(2\cos\frac{\beta}{2}\cdot\sin\frac{\delta}{2}\right)\\ &\geqslant 0.\end{aligned}$$

Hence

$$|AB - CD| \geqslant |AC - BD|,$$
$$|AD - BC| \geqslant |AC - BD|.$$

Summing up above two inequalities, we obtain inequality (1). □

There is a special inscribed quadrilateral, called double scribed quadrilateral, which means it has both the circumscribed and inscribed circles.

The following example is an inequality of a double scribed quadrilateral. The inequality was found by Mr. Chen Jixian. The following proofs (1) and (2) we introduce were provided by Long Yun (former student of Yali High School of Changsha, China, who was elected to the National Math Winter Campus of China in 1999) and Zhu Qingsan (student), respectively.

Example 2. Let $ABCD$ be a convex double scribed quadrilateral. Denote the radius and area of the circumcirle by R and S, respectively. Let a, b, c, d be the side lengths of the quadrilateral $ABCD$. Prove that

$$abc + abd + acd + bcd \leqslant 2\sqrt{S}(S + 2R^2). \tag{2}$$

Proof 1. We denote the centers of the circumcircle and the inscribed circle of quadrilateral $ABCD$ by O and I, respectively (see Figure 3.2). The tangent points of inscribed circle with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ are K, L, M, N, respectively. Let $\angle AIN = \angle 1$, $\angle BIK = \angle 2$, $\angle CIL = \angle 3$, $\angle DIM = \angle 4$, and denote $AK = AN = a'$, $BL = BK = b'$, $CL = CM = c'$, $DM = DN = d'$.

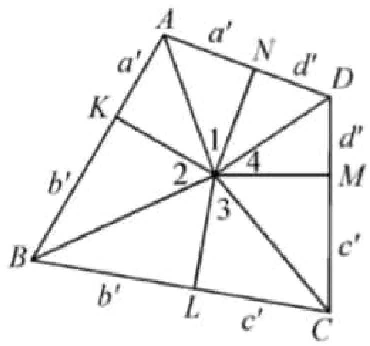

Figure 3.2

Since $ABCD$ has an inscribed circle, we have

$$a + c = b + d.$$

Denote the left side of (1) by H, and without loss of generality, we may assume that the radius of the inscribed circle is 1, it follows that

$$H = (a + c)bd + (b + d)ac = \frac{1}{2}(a + b + c + d)(ac + bd). \tag{a}$$

Applying

$$a = a' + b',\ b = b' + c',\ c = c' + d',\ d = d' + a',$$

to the right side of (a), yields

$$H = (a' + b' + c' + d')[(a' + b')(c' + d') + (b' + c')(d' + a')]. \tag{b}$$

Since $\angle A + \angle C = 180^\circ$, we obtain

$$\angle 1 + \angle 3 = 90^\circ.$$

From this, we have $\triangle AIN \backsim \triangle ICL$, hence

$$a'c' = AN \cdot CL = NI \cdot IL = 1. \tag{c}$$

Similarly

$$b'd' = 1. \tag{d}$$

Notice that

$$b'd' = 1. \tag{d}$$

Notice that

$$S = \frac{a+b+c+d}{2} \cdot r = a' + b' + c' + d', \tag{e}$$

from equalities (c), (d), (e), we obtain

$$H = S[4 + (a' + c')(b' + d')]. \tag{f}$$

On the other hand, from sine law and $\angle B + 2\angle 2 = 180°$, we have

$$\begin{aligned} R &= \frac{AC}{2\sin\angle B} \\ &= \frac{AC}{2\sin 2\angle 2} \\ &= \frac{AC}{4}\left(\tan\angle 2 + \frac{1}{\tan\angle 2}\right) \\ &= \frac{1}{4}AC(\tan\angle 2 + \tan\angle 4) \\ &= \frac{1}{4}AC \cdot (b' + d'). \end{aligned}$$

Similarly

$$R = \frac{1}{4}BD \cdot (a' + c'),$$

therefore

$$R^2 = \frac{1}{16}AC \cdot BD(a' + c')(b' + d'),$$

but

$$S = \frac{1}{2}AC \cdot BD \cdot \sin\alpha \leqslant \frac{1}{2}AC \cdot BD,$$

where α is the angle of diagonal AC and BD. Hence

$$R^2 \geqslant \frac{1}{8}S(a' + c')(b' + d').$$

From above we have

$$2\sqrt{S}(S + 2R^2) \geqslant 2\sqrt{S}\left(S + \frac{S}{4} + (a' + c')(b' + d')\right)$$

$$= \frac{S^{\frac{3}{2}}}{2}[4 + (a' + c')(b' + d')]. \tag{g}$$

By (f), (g), in order to prove (1), it suffices to prove $\frac{1}{2}S^{\frac{1}{2}} \geqslant 1$,

$$= \frac{S^{\frac{3}{2}}}{2}[4 + (a' + c')(b' + d')]. \quad \text{(g)}$$

By (f), (g), in order to prove (1), it suffices to prove $\frac{1}{2}S^{\frac{1}{2}} \geqslant 1$, which is equivalent to

$$\sqrt{a' + b' + c' + d'} \geqslant 2. \quad \text{(h)}$$

Since $a'c' = 1$, $b'd' = 1$, we obtain

$$a' + c' + b' + d' \geqslant 2\sqrt{a'c'} + 2\sqrt{b'd'} = 4,$$

so we have proved (h). □

The above natural and fluent proof that uses fine triangulation methods won high praise by lots of math olympic masters.

Proof 2. First we prove a lemma.

Lemma 1. In $\triangle ABC$, if $\angle A \geqslant 90°$, then $(b+c)/a \leqslant \sqrt{2}$.

Proof.

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = 2\frac{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A}$$
$$\leqslant \frac{2\cos\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{1}{\sin\frac{A}{2}} \leqslant \sqrt{2}.$$ □

Let us prove the original inequality.

Proof. Assuming that the four sides of quadrilateral $ABCD$ be AB, BC, CD, DA (see Figure 3.3), where the four sides lengths are a, b, c, d, respectively. And let the inscribed radius of quadrilateral $ABCD$ be 1. Since

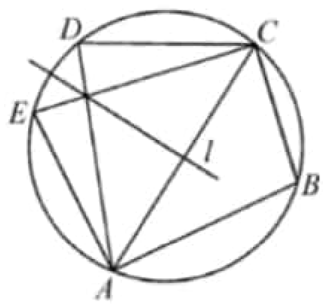

Figure 3.3

$$a + c = b + d = \frac{1}{2}(a + b + c + d) \cdot 1 = S,$$

we get

$$\begin{aligned} H &= abc + abd + acd + bcd \\ &= ac(b+d) + bd(a+c) = (ac+bd)S. \end{aligned} \tag{a}$$

Let the bisector of AC be l, D and E be symmetric to the line l, therefore

$$\triangle ACD \cong \triangle CAE,$$

thus $AE = c$, $CE = d$, and $\angle E = \angle D = \pi - \angle B$, it follows that A, E, C, B lie on a circle, so

$$S = \frac{1}{2}(ac+bd)\sin\alpha, \tag{b}$$

where $\alpha = \angle EAB$.

From (a), (b), we know that the inequality is equivalent to

$$\frac{2S}{2\sin\alpha} \cdot S \leqslant 2\sqrt{S}(S+2R^2). \tag{c}$$

Notice that $R = \dfrac{BE}{2\sin\alpha}$, so (c) is further equivalent to

$$S^{\frac{3}{2}} \leqslant S\sin\alpha + \frac{BE^2}{2\sin\alpha}, \tag{d}$$

according to the mean value inequality, we get

$$S\sin\alpha + \frac{BE^2}{2\sin\alpha} \geqslant 2\sqrt{\frac{S \cdot BE^2}{2}}.$$

To prove (d), it suffices to prove

$$\sqrt{2}BE \geqslant S. \tag{e}$$

In fact, since $\angle EAB + \angle ECB = 180^\circ$, assuming that $\angle EAB \geqslant 90^\circ$. Applying Lemma 1 to $\triangle ABE$, we get $\dfrac{a+c}{BE} \leqslant \sqrt{2}$, and it follows that

$$\sqrt{BE} \geqslant a + c = S,$$

so we have proved (e). □

The method above used trigonometric function and geometry, and constructed a new inscribed quadrilateral, so transformed the problem.

The inscribed quadrilateral has a famous extremal property: Of all quadrilaterals with given sides, the inscribed quadrilateral has the maximum area.

Theorem 1. Let the four sides of the convex inscribed quadrilateral be a, b, c, d, respectively, and s be half of the perimeter, then the area F of the quadrilateral is

$$F=\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{3}$$

If $d=0$, this is the Heron's area formula for triangular. The proof below quotes from the book-Modern Geometry by Roger A. Johnson. (Translated by Shan Zun, Shanghai Education Publishing House, 1999.)

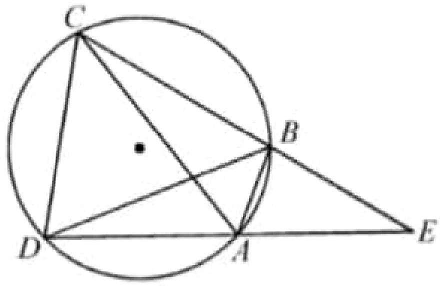

Figure 3.4

Proof. Let the quadrilateral be $ABCD$, and $AB=a$, $BC=B$, $CD=c$, $DA=d$, see Figure 3.4.

If the quadrilateral is a rectangle, the proof is obvious. Otherwise, we assume AD and BC are extended to intersect at point E outside the circle, and $CE=x$, $DE=y$, according to the area formula of the triangle, we obtain the area of $\triangle CDE$

$$S_{\triangle CDE}=\frac{1}{4}\sqrt{(x+y+c)(x+y-c)(x-y+c)(-x+y+c)}. \tag{a}$$

Note that $\triangle ABE \backsim \triangle CDE$, so we have

$$\frac{S_{\triangle ABE}}{S_{\triangle CDE}}=\frac{a^2}{c^2},$$

therefore

$$\frac{F}{S_{\triangle CDE}} = \frac{c^2 - a^2}{c^2}, \tag{b}$$

and from the proportion relations

$$\frac{x}{y} = \frac{y - d}{a},$$

$$\frac{y}{c} = \frac{x - b}{a},$$

we obtain

$$x + y + c = \frac{c}{c - a}(-a + b + c + d),$$

similarly, we can get expressions of $x + y - c$ etc..

Substitute them to (a) and simplifying, we have

$$S_{\triangle CDE} = \frac{c^2}{c^2 - a^2}\sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{c}$$

substitute (c) into (b), we obtain the inequality (3). □

Generalization. Let the side lengths of a convex quadrilateral be a, b, c and d, respectively, and the sum of opposite angles be $2u$. Prove that the area F can be given by

$$F^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2 u.$$

The proof of it is too tedious and insipid to be given here. But we can see from it that if the four sides of a quadrilateral are given, the inscribed quadrilateral has the maximum area.

The example below uses the extremal property of the inscribed quadrilateral.

Example 3 (Popa's inequality). If the convex quadrilateral with the area F and four sides satisfying $a \leqslant b \leqslant c \leqslant d$.

Prove that

$$F \leqslant \frac{3\sqrt{3}}{4}c^2. \tag{4}$$

Proof. By the extremal property of the inscribed quadrilateral, we only need to prove (4) for inscribed quadrilateral.

$$F^2 = (s-a)(s-b)(s-c)(s-d),$$

where $s = \frac{1}{2}(a+b+c+d)$, $s-d = (a+b+c)-s$. By the arithmetic-geometric inequality, we have

$$\begin{aligned} F^2 &= 3^3\left(\frac{1}{3}s - \frac{1}{3}a\right)\left(\frac{1}{3}s - \frac{1}{3}b\right)\left(\frac{1}{3}s - \frac{1}{3}c\right)(a+b+c-s) \\ &\leqslant 3^3\left[\frac{\left(\frac{1}{3}s - \frac{1}{3}a\right) + \left(\frac{1}{3}s - \frac{1}{3}b\right) + \left(\frac{1}{3}s - \frac{1}{3}c\right) + (a+b+c-s)}{4}\right]^4 \\ &= 3^3\left(\frac{a+b+c}{3\cdot 2}\right)^4 \leqslant 3^3\left(\frac{c}{2}\right)^4, \end{aligned}$$

and the last step uses $a \leqslant b \leqslant c$.

Thus

$$F \leqslant \frac{3\sqrt{3}}{4}c^2.$$

□

The following is another typical problem.

Example 4 (Gaolin's inequality). Let the convex quadrilaterals $ABCD$ and $A'B'C'D'$ have side lengths a, b, c, d and a', b', c', d', and with the area F, F', respectively.

Denote

$$K = 4(ad+bc)(a'd'+b'c') - (a^2-b^2-c^2+d^2)(a'^2-b'^2-c'^2+d'^2).$$

Prove that

$$K \geqslant 16FF'. \tag{5}$$

Proof. By the extremal property of the inscribed quadrilateral, it

suffices to consider the inscribed quadrilaterals, see Figure 3.5. Since $\angle B + \angle D = 180°$, we have

$$2F = (ad + bc)\sin B, \qquad \text{(a)}$$

similarly

$$2F' = (a'd' + b'c')\sin B'. \qquad \text{(b)}$$

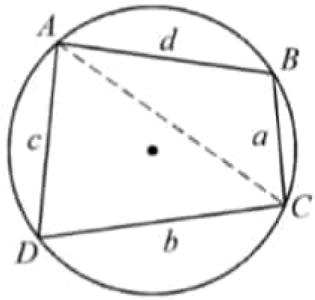

Figure 3.5

On the other hand, by the cosine law, we obtain

$$AC^2 = b^2 + c^2 + 2bc\cos B = a^2 + d^2 - 2ad\cos B,$$

therefore

$$a^2 - b^2 - c^2 + d^2 = 2(ad + bc)\cos B, \qquad \text{(c)}$$

likewise

$$a'^2 - b'^2 - c'^2 + d'^2 = 2(a'd' + b'c')\cos B', \qquad \text{(d)}$$

from relations (a) and (d), we have

$$K - 16FF' = 4(ad + bc)(a'd' + b'c')(1 - \cos(B - B')) \geqslant 0,$$

as desired. □

Remark. By the above proof we can write the inequality even more general

$$0 \leqslant K - 16FF' = 8(ad + bc)(a'd' + b'c'),$$

and the left side of the above inequality is Gaolin's inequality.

Gaolin's inequality can be regarded as the generalization of Neuberg-Pedoe's inequality for quadrilateral.

At the end of this section, we study a much harder extremal problem of the inscribed quadrilateral. The solution is provided by Xiang Zhen (former student of the First High School of Changsha City, China, who won a gold medal at the 44th IMO).

Example 5. For given radius R and area ($S \leqslant 2R^2$) of the circumcircle for a double scribed quadrilateral $ABCD$. Evaluate the

maximal value of plm, where p is the semi-perimeter of the quadrilateral, l and m are the lengths of the two diagonals.

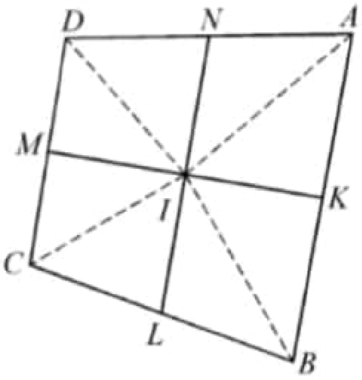

Figure 3. 6

Answer. Let r be the radius of the inscribed circle, $\alpha = \angle AIK$, $\beta = \angle BIK$, I be the circumcenter of the quadrilateral $ABCD$, and K be the tangent point of $\odot I$ and line AB, see Figure 3.6.

Since the semi-perimeter of the quadrilateral

$$p = r(\tan\alpha + \cot\alpha + \tan\beta + \cot\beta) = r\left(\frac{2}{\sin 2\alpha} + \frac{2}{\sin\beta}\right),$$

it follows that

$$S = rp = r^2\left(\frac{2}{\sin 2\alpha} + \frac{2}{\sin 2\beta}\right). \tag{a}$$

In triangle ABD, we have

$$AB = r(\tan\alpha + \tan\beta),\ AD = r(\tan\alpha + \cot\beta),\ \angle DAB = \pi - 2\alpha.$$

By the cosine law yields

$$\begin{aligned} BD^2 &= r^2[(\tan\alpha + \tan\beta)^2 + (\tan\alpha + \cot\beta)^2 \\ &\quad + 2\cos 2\alpha(\tan\alpha + \tan\beta)(\tan\alpha + \cot\beta)] \\ &= r^2\left(\tan\alpha \cdot \frac{2}{\sin 2\beta} \cdot 4\cos^2\alpha + \frac{4}{\sin^2 2\beta}\right). \end{aligned}$$

Therefore

$$R^2 = \frac{BD^2}{4\sin^2 2\alpha} = r^2\left(\frac{1}{\sin 2\alpha \sin 2\beta} + \frac{1}{\sin^2 2\alpha \sin^2 2\beta}\right). \tag{b}$$

Denote $a = \sin 2\alpha$, $b = \sin 2\beta$, we get a, $b \in (0, 1]$, so (a), (b), can be written as

$$S = 2r^2\,\frac{a + b}{ab}, \tag{c}$$

$$R^2 = r^2\,\frac{1 + ab}{a^2 b^2}, \tag{d}$$

divided (d) by (c) yields

$$\frac{ab(a+b)}{1+ab}=\frac{S}{2R^2}. \tag{e}$$

(e) is the constraint condition of a, b. By this condition, we deduce the maximal value. Since

$$p=r\cdot\left(\frac{2}{a}+\frac{2}{b}\right),$$
$$lm=4R^2ab$$

we have

$$(plm)^2=64R^4r^2(a+b)^2=16R^2S^2(1+ab),$$

therefore

$$plm=4RS\sqrt{1+ab}. \tag{f}$$

From (e), we obtain

$$\frac{S}{2R^2}=\frac{ab(a+b)}{1+ab}\geqslant\frac{ab\cdot 2\sqrt{ab}}{1+ab}. \tag{g}$$

Denote $\sqrt{ab}=x$, with $x\in(0,1]$, (g) gives

$$4R^2\cdot x^3-S\cdot x^2-S\leqslant 0. \tag{h}$$

Define function $f(x)=4R^2\cdot x^3-S\cdot x^2-S$, notice that

$$f(0)=-S\leqslant 0,\ f(1)=4R^2-2S\geqslant 0$$

and

$$\begin{cases} f'(x)\geqslant 0,\ x\geqslant\dfrac{S}{6R^2}, \\ f'(x)<0,\ 0<x<\dfrac{S}{6R^2}. \end{cases}$$

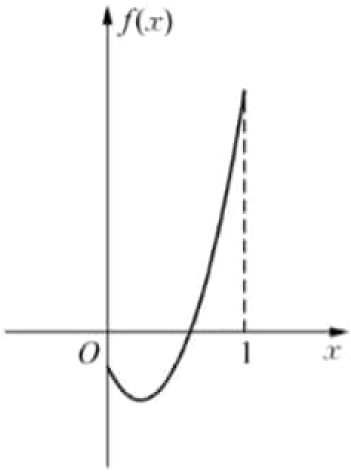

Figure 3.7

Thus $f(x)$ decreasing first and then increasing. Therefore $f(x)$ has a unique root in $(0, 1)$, see Figure 3.7.

By (h) we see that $f(x)\leqslant 0$, therefore,

$$\sqrt{ab} \leqslant t,$$

then $ab \leqslant t^2$. Applying it to (f), we obtain

$$plm = 4RS\sqrt{1+ab} \leqslant 4RS\sqrt{1+t^2}.$$

If $a = b = t$, the equality holds, thus the maximal of plm is $4RS\sqrt{1+t^2}$, and t is the root of $4R^2x^3 - Sx^2 - S = 0$ in interval $(0, 1]$.

Exercises 3

1. Let $ABCD$ be an inscribed convex quadrilateral with interior angles and exterior angles no less than 60°. Prove that

$$\frac{1}{3}\left|AB^3 - AD^3\right| \leqslant \left|BC^3 - CD^3\right| \leqslant 3\left|AB^3 - AD^3\right|,$$

and point out the condition such that the equality holds.

2. Let $ABCD$ be a convex circumscribed quadrilateral $ABCD$ with area S and the circumcenter is inside the quadrilateral. The intersection point of two diagonals is denoted by E, and let M, N, P, Q be the projection of E on four sides, respectively. Prove that the area of $MNPQ$ is no more than $\frac{S}{2}$.

3. Let a, b, c and d and a', b', c', d' be the side lengths, S and S' be the areas of two convex quadrilaterals $ABCD$ and $A'B'C'D'$, respectively. Prove that $aa' + bb' + cc' + dd' \geqslant 4\sqrt{SS'}$.

4. (An Zhenping) Let $ABCD$ be an circumscribed quadrilateral with side lengths a, b, c, d. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2d(c-d) + d^2a(d-a) \geqslant 0.$$

5. (Groenman) Let $ABCD$ be an inscribed quadrilateral with side lengths a, b, c, d. And ρ_a is the radius of the circle outside the quadrilateral, and tangent to the edges AB, CB, and extended line

DA. The ρ_b, ρ_c, ρ_d are defined similarly. Prove that

$$\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}+\frac{1}{\rho_d}\geqslant\frac{8}{\sqrt[4]{abcd}}$$

and the equality holds if and only if the $ABCD$ is a square.

Chapter 4 The area inequality for special polygons

The area inequalities and extreme value problem for polygons have attracted much attention.

Some area inequalities for special polygons such as parallelogram and triangle often appeared at middle school math competitions. In this section, we introduce some interesting results, and try our best to treat the area problems more generally.

First, we look into the relationship of area between the parallelogram and the inscribed triangle in it. A well-known conclusion of it is: any area of the inscribed triangle does not exceed the half of the parallelogram area.

The proof of this conclusion is quite simple, see Figure 4.1, just make line passing point Q, the apex of the triangle PQR, paralleling to AB, and consider the relations between the areas of the small parallelogram and the triangle it contains.

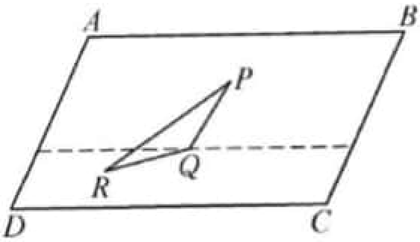

Figure 4.1

Now consider the opposite problem: what is the relationship of the areas between the parallelogram and the triangle within? The answer is a useful theorem as follows.

Theorem 1. The area of a parallelogram in any triangle is no more than half area of the triangle.

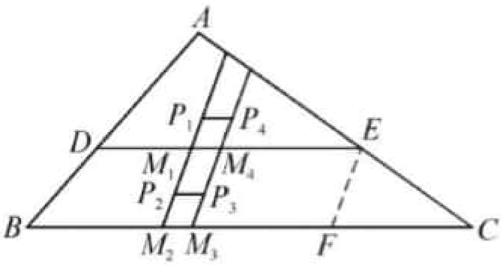

Figure 4.2

Proof. Considering the parallelogram

$P_1P_2P_3P_4$ contained in a triangle ABC, see Figure 4.2. Let M_2, M_3 be the intersection points of lines P_1P_2, P_3P_4 with BC, respectively. Assume that $M_2M_1 = P_2P_1$, $M_3M_4 = P_3P_4$, and $M_1 \in P_1P_2$, $M_4 \in P_3P_4$, so $M_1M_2M_3M_4$ is a parallelogram, and

$$S(M_1M_2M_3M_4) = S(P_1P_2P_3P_4).$$

Let the intersection point of the line M_1M_4 with AB and AC be D and E, respectively. Let EF be parallel to AB, so that $BDEF$ is a parallelogram, hence

$$S(BDEF) \geqslant S(M_1M_2M_3M_4) = S(P_1P_2P_3P_4).$$

If we want to prove

$$S(P_1P_2P_3P_4) \leqslant \frac{1}{2}S_{\triangle ABC},$$

we must prove

$$S(BDEF) \leqslant \frac{1}{2}S_{\triangle ABC}. \tag{a}$$

Now we proceed to prove (a).

Denote $\lambda = \dfrac{AD}{AB}$, see Figure 4.3. Since

$$\triangle ADE \backsim \triangle ABC,$$

we obtain

$$S_{\triangle ADE} = \lambda^2 S_{\triangle ABC}.$$

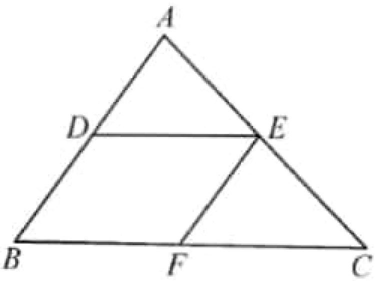

Figure 4.3

Similarly,

$$S_{\triangle EFC} = (1-\lambda)^2 S_{\triangle ABC}.$$

Therefore,

$$S_{\triangle ADE} + S_{\triangle EFC} = [\lambda^2 + (1-\lambda)^2]S_{\triangle ABC} \geqslant \frac{1}{2}S_{\triangle ABC}.$$

Thus

$$S(BDEF) = S_{\triangle ABC} - (S_{\triangle ADE} + S_{\triangle EFC}) \leqslant \frac{1}{2} S_{\triangle ABC},$$

as desired, and equality holds if and only if D, E, F are the midpoints. □

The method above is a typical method of transformation, that is to say, the parallelogram $P_1P_2P_3P_4$ is transformed into parallelogram $M_1M_2M_3M_4$ of which a pare sides are parallel to the side BC, and then transformed into a special parallelogram $BDEF$ whose two pare sides are parallel to the sides of triangle respectively, thus the problem has been greatly simplified.

Let P be a point in $\triangle ABC$, see Figure 4.4, and D, E, F be the intersection points of lines AP, BP, CP with the three sides, respectively. The $\triangle DEF$ is called Ceva's triangle to P.

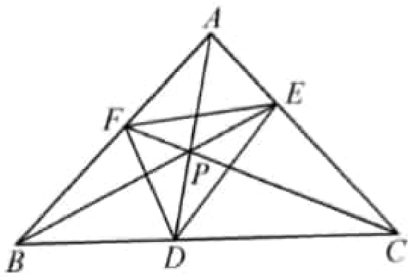

Figure 4.4

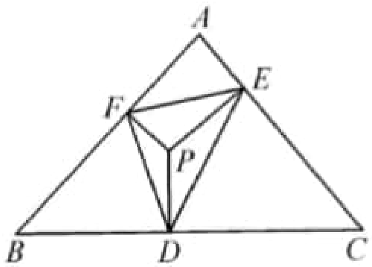

Figure 4.5

Let P be a point in $\triangle ABC$, see Figure 4.5, and D, E and F be the projection of P onto BC, CA and AB, respectively. The $\triangle DEF$ is called the pedal triangle to P.

The following are famous theorems on Ceva's triangle and the pedal triangle.

Proposition 7. Let P be the point in $\triangle ABC$, then the area of Ceva's triangle $\triangle DEF$ to P is not more than $\frac{1}{4} S_{\triangle ABC}$.

Proposition 8. Let P be the point in $\triangle ABC$, then the area of triangle of pedal triangle $\triangle DEF$ to P is not more than $\frac{1}{4} S_{\triangle ABC}$.

Mr. Yang Lin noticed the relationship between Theorem 1 and Proposition 7, and found that the Proposition 7 is the corollary of Theorem 1 by the expansion of Ceva's $\triangle ABC$ as follows.

Example 1. Let P be the point in $\triangle ABC$, and $\triangle DEF$ be Ceva's triangle to P. Show that in $\triangle ABC$ there is a parallelogram with two sides of $\triangle DEF$ as its adjacent sides.

Proof. Let G be the orthocenter of $\triangle ABC$, N, M be the midpoints of the sides AC and AB, respectively, see Figure 4.6. Without loss of generality, we assume that P is at the side or interior of $ANGM$, so E, F lie on segment AN, AM or on endpoints of them, and

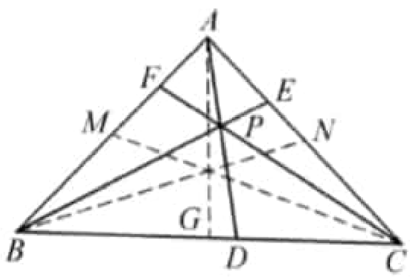

Figure 4.6

$$\frac{AF}{FB} \leqslant 1, \ \frac{AE}{EC} \leqslant 1,$$

and without loss of generality, we assume that

$$\frac{AF}{FB} \leqslant \frac{AE}{EC}.$$

By Ceva's theorem, we obtain

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

therefore

$$\frac{BD}{DC} = \frac{AE}{CE} \cdot \frac{FB}{AF} \geqslant 1.$$

We can construct the parallelogram $FEDE'$ with adjacent sides EF and ED, see Figure 4.7. So we need only to show that E' lies in the interior or on the side of $\triangle ABC$.

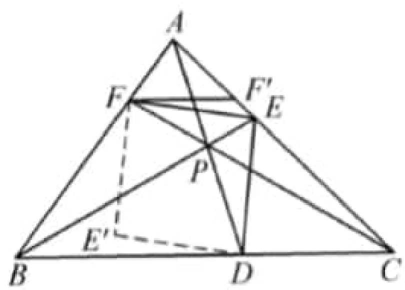

Figure 4.7

Draw a line FF' parallel to BC, so F' lies on AC. And since

$$\frac{AF}{FB} \leqslant \frac{AE}{EC},$$

it follows that F' lies on segment AE or endpoints. Because

$$\angle E'DF = \angle EDF \leqslant \angle F'FD = \angle FDB,$$

we see that DE' lies in the interior of $\angle FDB$.

Likewise

$$\frac{CE}{EA} \geqslant 1 \geqslant \frac{CD}{DB},$$

and we can also show that FE' lies in the interior or on the side of $\angle BDF$, thus E' lies in the interior of $\triangle FDB$ as desired. □

Theorem 1 and Proposition 7 are linked by Example 1, that is to say

$$\text{Theorem 1} \Rightarrow \text{Proposition 7}.$$

A natural question is, does the pedal triangle of the inner point P have a similar extension property as Ceva's triangle?

It is easy to see that the pedal triangle about the inner point in obtuse triangle does not have the extension property generally, but the answer is positive to the acute triangle.

Example 2. Let P be the interior point of acute triangle $\triangle ABC$, $\triangle DEF$ be the pedal triangle about P. Show that in $\triangle ABC$ there is a parallelogram with two sides of $\triangle DEF$ as its adjacent sides.

Proof. Let O be the circumcenter of $\triangle ABC$. Since $\triangle ABC$ is acute, O lies in $\triangle ABC$. Without loss of generality we may assume that P lies in $\triangle AOB$, see Figure 4.8.

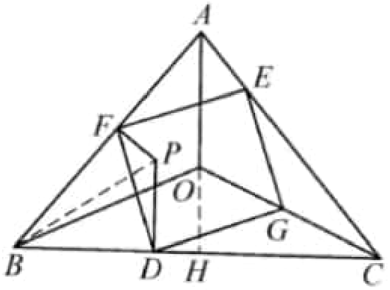

Figure 4.8

To prove the parallelogram $DFEG$ with

area is no more than $\frac{1}{4}$. The proof is too long to be given here.

2. Given an arbitrary graph F, let S_F be the smallest positive integer n satisfying the following conditions, in the internal of F (including the boundary), given arbitrary n points, there exist three points, the area of triangle constituted by them does not exceed $\frac{|F|}{4}$, where $|F|$ indicates the area of F. A. Soifer's problem is equivalent to the following proposition.

Proposition 9. For any triangle F, $S_F = 5$.

A. Soifer further proved the following:

Proposition 10. For any parallelogram F, $S_F = 5$.

A natural question is this: whether $S_F = 5$ holds for any of the graphics F or not?

The answer is negative, A. Soifer proved the following:

Proposition 11. For regular pentagon, $S_F = 6$.

For any graphics F, what value can S_F attain? A. Soifer had proved that S_F can only take in a very small range.

Proposition 12. For convex graphics, $4 \leqslant S_F \leqslant 6$.

The further improvements of Proposition 12 are as follows:

Proposition 13. For convex graphics F, $S_F \neq 4$.

Proposition 14. For convex graphics F, $S_F = 5$, or $S_F = 6$.

However, an interesting open question is: What kind of convex graphics F such that $S_F = 5$, and what kind of convex graphics F such that $S_F = 6$?

The following discussion is about what kind of parallelogram or triangle can cover the convex polygon with area 1. We have:

Example 4. (1) The convex polygon with area 1 can be covered by parallelogram with area 2. (2) The convex polygon with area 1 can be covered by triangle with area 2.

Proof. (1) First let a convex polygon M with area 1 be on one side of the support line AB, there exists one point C in M with the greatest distance to AB, C may be a vertex of M or lies in the line parallel to AB. Connect AC, see Figure 4.12, and it divides M into two parts M_1, M_2 (if AC is a side of M, there is no M_1, or M_2). Assume that points D_1 and D_2 are the farthest points to line AC on M, and located at two sides of AC. Draw a straight line parallel to AB through C, and straight lines l_1, l_2 parallel to AC, so straight lines AB, l, l_1 and l_2 constitute a parallelogram P containing M.

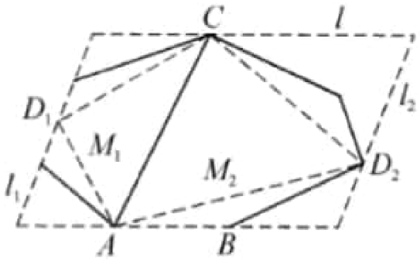

Figure 4.12

Since M_1 and M_2 are convex, they contain $\triangle AD_1C$, $\triangle AD_2C$.

Suppose that P divided by AC into two parallelograms P_1 and P_2, so

$$S_{\triangle AD_1C} = \frac{1}{2}S(P_1),\ S_{\triangle AD_2C} = \frac{1}{2}S(P_2)$$

with $S(X)$ being the area of X, therefore

$$\begin{aligned} S(P) &= S(P_1) + S(P_2) \\ &= 2S_{\triangle AD_1C} + 2S_{\triangle AD_2C} \\ &\leqslant 2S(M_1) + 2S(M_2) = 2S(M) = 2 \end{aligned}$$

as desired.

(2) Let u be a given polygon with area 1, now we consider the internal triangle $\triangle A_1A_2A_3$ with largest area. We discuss it in two cases.

(a) If $\triangle A_1A_2A_3 \leqslant 1/2$. See Figure 4.13, draw three straight lines passing vertexes of $\triangle A_1A_2A_3$ and parallel to the opposite sides, respectively. These three lines constitute a triangle, denote by T, so the area of T is 2.

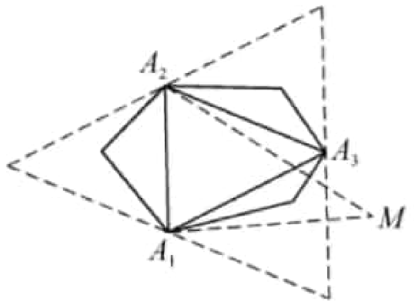

Figure 4.13

Therefore we need only to prove that polygon u lies in T. Suppose that some point M of u were out of T, then the distance of M to some side of $\triangle A_1A_2A_3$ would be greater than that of the vertex A_3 of $\triangle A_1A_2A_3$ to this side. Without loss of generality, assume this side were A_1A_2, see Figure 4.13, in this case the area of $\triangle A_1A_2M$ would be greater than that of $\triangle A_1A_2A_3$. That is contrary to the fact that $\triangle A_1A_2A_3$ has the maximum area in u.

(b) If $\triangle A_1A_2A_3 > 1/2$. There are three parts of u but outside of $\triangle A_1A_2A_3$, see Figure 4.14. In each part, construct triangles with the largest area and one side of $\triangle A_1A_2A_3$ as the base. Denote these triangles by $\triangle B_1A_2A_3$, $\triangle B_2A_1A_3$ and $\triangle B_3A_1A_2$, respectively, then draw lines through B_1, B_2 and B_3 and parallel to A_2A_3, A_1A_3 and A_1A_2, respectively. Thus, we obtain a larger triangle $\triangle C_1C_2C_3$, denoted by C. We can prove that u lies in triangle C as case (a).

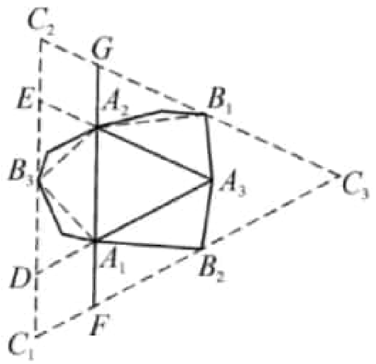

Figure 4.14

Note that u is a convex polygon, then

$$S(A_1B_3A_2B_1A_3B_2) \leqslant S(u) = 1.$$

So we need only to prove that

$$S_{\triangle C_1C_2C_3} \leqslant 2S(A_1B_3A_2B_1A_3B_2). \tag{1}$$

Since $\triangle C_1C_2C_3 \backsim \triangle A_1A_2A_3$, in order to calculate the area of $\triangle C_1C_2C_3$, we denote

$$\frac{S_{\triangle A_1A_2B_3}}{S_{\triangle A_1A_2A_3}} = \lambda_3,\ \frac{S_{\triangle A_1A_3B_2}}{S_{\triangle A_1A_2A_3}} = \lambda_2,\ \frac{S_{\triangle A_2A_3B_1}}{S_{\triangle A_1A_2A_3}} = \lambda_1,$$

it follows that

$$\frac{S_{\triangle C_1C_2C_3}}{S_{\triangle A_1A_2A_3}} = (\lambda_1 + \lambda_2 + \lambda_3 + 1)^2. \tag{2}$$

By hypothesis $S_{\triangle A_1A_2B_3} \geqslant 1/2$, we have

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \frac{S_{\triangle A_1A_2B_3} + S_{\triangle A_1A_3B_2} + S_{\triangle A_2A_3B_1}}{S_{\triangle A_1A_2A_3}} \\ &\leqslant \frac{S(u) - S_{\triangle A_1A_2A_3}}{S_{\triangle A_1A_2A_3}} \\ &= \frac{1 - S_{\triangle A_1A_2A_3}}{S_{\triangle A_1A_2A_3}} \\ &< 1,\end{aligned} \tag{3}$$

and

$$\begin{aligned}\frac{S(A_1B_3A_2B_1A_3B_2)}{S_{\triangle A_1A_2A_3}} &= \frac{S_{\triangle A_1A_2A_3} + S_{\triangle B_1A_2A_3} + S_{\triangle B_2A_1A_3} + S_{\triangle B_3A_1A_2}}{S_{\triangle A_1A_2A_3}} \\ &= \lambda_1 + \lambda_2 + \lambda_3 + 1.\end{aligned} \tag{4}$$

By (2), (3), (4), we obtain

$$\frac{S_{\triangle C_1C_2C_3}}{S(A_1B_3A_2B_1A_3B_2)} = \lambda_1 + \lambda_2 + \lambda_3 + 1 < 2.$$

Thus (1) holds, as desired. □

Now we consider another interesting question: what is the largest area of a triangle inscribed in a convex polygon with area 1? The following examples partly give the answer.

Example 5. (1) Let M be a convex polygon with area 1, and l an arbitrary given line. Prove that there exists a triangle inscribed M with one side parallel to l and area greater than or equal to $\frac{3}{8}$.

(2) Let M be a regular hexagon with area 1, and l be an arbitra given line. Prove that there does not exist inscribed triangle in M w one side parallel to l and area greater than $\frac{3}{8}$.

Proof. (1) As Figure 4.15 shows, draw two supporting lines l_1, l_2 of M parallel to l so that M lies in the zonal region and the vertexes A and B on the parallel lines. Let the width between l_1 and l_2 be d, draw three straight lines l'_1, l_0, l'_2, divide the zonal region into four small strips with the same width $\frac{1}{4}d$. Assume the boundary of M intersect l'_1 at points P and Q, and intersect l'_2 at points R and S. (Since M is convex, its side cannot entirely lie on a straight line.)

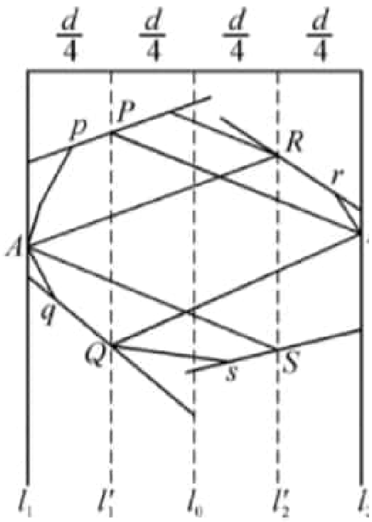

Figure 4.15

Denote p the line on which the side of M passing the point P lies, (i is the vertex, we can choose any adjacent side). Denote r anc similarly. The trapezoid constructed by lines p, q, l_0 and l_1 has a $\frac{d}{2}\cdot RQ$. Similarly, the trapezoid constructed by lines r, r, l_0 and l_2 area $\frac{d}{2}\cdot RS$. Since the union set of T_1 and T_2 contains M, so we have

$$S(M)\leqslant S(T_1)+S(T_2)=\frac{d}{2}\cdot PQ+\frac{d}{2}\cdot RS=\frac{d}{2}PQ+RS.$$

Now we consider two triangles $\triangle ARS$ and $\triangle BPQ$, and we fi that they are both triangles inscribed in M, and

$$S_{\triangle ARS}=\frac{1}{2}\cdot RS\cdot\frac{3}{4}d,\ S_{\triangle BPQ}=\frac{1}{2}\cdot PQ\cdot\frac{3}{4}d,$$

therefore

$$S_{\triangle ARS} + S_{\triangle BPQ} = (PQ + RS) \cdot \frac{3}{8}d = \frac{3}{4}(PQ + RS) \cdot \frac{1}{2}d$$
$$\geqslant \frac{3}{4}S(M) = \frac{3}{4},$$

so at least one of the following inequalities holds:

$$S_{\triangle ARS} \geqslant \frac{3}{8},\ S_{\triangle BPQ} \geqslant \frac{3}{8}.$$

(2) Let M be a regular hexagon $ABCDEF$, and $l \parallel AB$, see Figure 4.16. Let $\triangle PQR$ have the largest area inscribed in M, and $PQ \parallel AB$. Without loss of generality, we assume that P and Q be in FA and BC, respectively, it follows that R must be in DE. Let the sides of regular hexagon equal one, and write $AP = BQ = a$, so we obtain

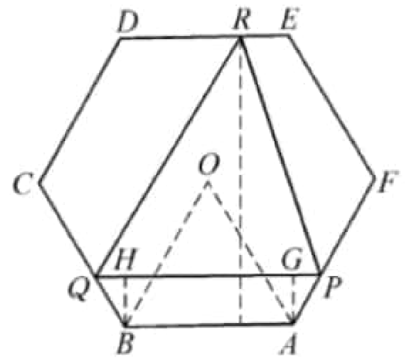

Figure 4.16

$$PQ = AB + PG + QH$$
$$= 1 + \frac{a}{2} + \frac{a}{2} = 1 + a,$$

and

$$h(PQR) = RS - AG = \sqrt{3} - \frac{a\sqrt{3}}{2}$$
$$= (2 - a)\frac{\sqrt{3}}{2},$$

thus

$$S_{\triangle PQR} = \frac{1}{2}(1 + a)(2 - a)\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{4}(2 + a - a^2)$$
$$= \frac{\sqrt{3}}{4}\left(2 + \frac{1}{4} - \left(a - \frac{1}{2}\right)^2\right).$$

From this we see that if $a = 1/2$, the area of $S_{\triangle PQR}$ is the largest, and

$$(S_{\triangle PQR})_{\max} = \frac{9\sqrt{13}}{16},$$

but the area of regular hexagon is

$$6S(OAB) = 6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2},$$

with O the center of regular hexagon.

This shows that the largest area of triangle inscribed in M with one side parallel to a given straight line l is $\frac{3}{8}S(M)$, so the claim holds. □

Exercises 4

1. Let the circumradius of an obtuse $\triangle ABC$ be 1. Prove that $\triangle ABC$ can be covered by an isosceles triangle with hypotenuse length $\sqrt{2}+1$.

2. If a convex polygon M cannot cover any triangle with area 1, prove that M can be covered by a triangle with area 4.

3. (Li Shijie) Let D, E, F be points on sides BC, CA, AB of $\triangle ABC$ respectively, different from vertexes A, B, C. Denote the area of $\triangle ABC$, $\triangle AEF$, $\triangle BDF$, $\triangle CDE$, $\triangle DEF$, by S, S_1, S_2, S_3, S_0, respectively. Prove that

$$S_0 \geqslant 2\sqrt{\frac{S_1S_2S_3}{S}},$$

the equality holds if and only if AD, BE, CF intersect at a point in $\triangle ABC$.

4. Show that, it is impossible to put two non-overlapping squares with side length more than $\sqrt{\frac{2}{3}}$ in a square with side length one.

5. Given any n points on plane, and any three of them can be formed a triangle. Let u_n be the ratio of largest area to the smallest area of the triangles, find the minimal value of u_5.

Chapter 5 Linear geometric inequalities

Many linear geometric inequalities give us the impression: simple but unusual, easy to be remembered. The proof of them is either ordinary or difficult. Most linear geometric inequalities in math contests are full of challenge.

Erdös-Mordell's inequality is the most famous one of linear geometric inequalities which we introduce here first.

Example 1 (Erdös-Mordell's inequality). Suppose that point P is in $\triangle ABC$. Let $PD = p$, $PE = q$, and $PF = r$ be distances from P to the sides BC, CA, and AB, respectively. Let $PA = x$, $PB = y$, $PC = z$, then

$$x + y + z \geqslant 2(p + q + r). \tag{1}$$

The equality holds if and only if $\triangle ABC$ is equilateral and P is its center.

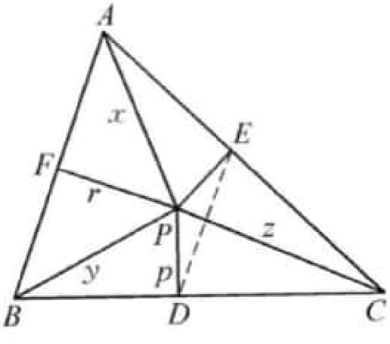

Figure 5.1

The following are five proofs to the inequality. Proof 1 is simple and widely cited, which was given by L. J. Mordell in 1937.

Proof. 1. Since $\angle DPE = 180° - \angle C$ (see Figure 5.1), by the cosine law, we get

$$\begin{aligned} DE &= \sqrt{p^2 + q^2 + 2pq\cos C} \\ &= \sqrt{p^2 + q^2 + 2pq\sin A\sin B - 2pq\cos A\cos B} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{(p\sin B + q\sin A)^2 + (p\cos B - q\cos A)^2} \\
&\geqslant \sqrt{(p\sin B + q\sin A)^2} \\
&= p\sin B + q\sin A.
\end{aligned}$$

Since P, D, C, E are on a circle, the line segment CP is the diameter of the circle, so

$$z = \frac{DE}{\sin C} \geqslant \left(\frac{\sin B}{\sin C}\right)p + \left(\frac{\sin A}{\sin C}\right)q,$$

similarly,

$$x \geqslant \left(\frac{\sin B}{\sin A}\right)r + \left(\frac{\sin C}{\sin A}\right)q,$$
$$y \geqslant \left(\frac{\sin A}{\sin B}\right)r + \left(\frac{\sin C}{\sin B}\right)p.$$

Adding above three inequalities together, we get

$$\begin{aligned}
x + y + z &\geqslant \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right)p + \left(\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A}\right)q + \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right)r \\
&\geqslant 2(p + q + r).
\end{aligned}$$

□

The following Proof 2 is given by Mr. Zhang Jingzhong, who applied the method of area subtly and concisely.

Proof. 2. Make MN through P such that $\angle AMN = \angle ACB$, then $\triangle AMN \backsim \triangle ACB$. (See Figure 5.2.)

We have

$$\frac{AN}{MN} = \frac{c}{a}, \ \frac{AM}{MN} = \frac{b}{a}.$$

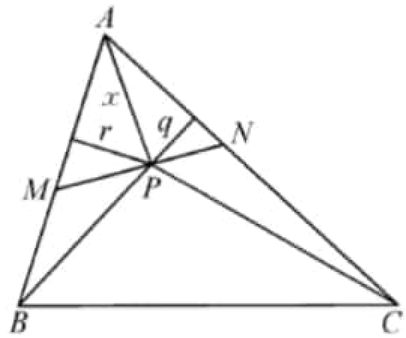

Figure 5.2

Since

$$S_{\triangle AMN} = S_{\triangle AMP} + S_{\triangle ANP},$$

we have

$$AP \cdot MN \geqslant q \cdot AN + r \cdot AM.$$

So that

$$x = AP \geqslant q \cdot \frac{AN}{MN} + r \cdot \frac{AM}{MN}.$$

Namely

$$x \geqslant \frac{c}{a} \cdot q + \frac{b}{a} \cdot r. \tag{a}$$

Similarly

$$y \geqslant \frac{a}{b} \cdot r + \frac{c}{b} \cdot p, \tag{b}$$

$$z \geqslant \frac{b}{c} \cdot p + \frac{a}{c} \cdot q. \tag{c}$$

Adding up inequalities (a), (b), (c), we get

$$\begin{aligned} x + y + z &\geqslant p\left(\frac{c}{b} + \frac{b}{c}\right) + q\left(\frac{c}{a} + \frac{a}{c}\right) + r\left(\frac{b}{a} + \frac{a}{b}\right) \\ &\geqslant 2(p + q + r). \end{aligned}$$

□

The following method of symmetric point has been noticed by lots of people. Here we adopt Mr. Zou Ming's proof, which is concise and comprehensible.

Proof. 3. Let the point P' and P be symmetric to the bisect of $\angle A$ (see Figure 5.3), then the distances from P' to CA, AB is r, q, respectively, and $P'A = PA = x$.

Let the distance from A, P' to BC be h_1, r'_1 respectively, then

$$P'A + r'_1 = PA + r'_1 \geqslant h_1,$$

multiply by a on both sides, we have

$$a \cdot PA + ar'_1 \geqslant ah_1$$

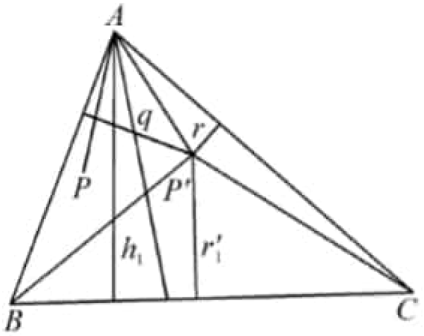

Figure 5.3

$$= 2S_{\triangle ABC}$$
$$= ar'_1 + cq + br.$$

So that

$$x \geqslant \frac{c}{a} \cdot q + \frac{b}{a} \cdot r,$$

similarly

$$y \geqslant \frac{a}{b} \cdot r + \frac{c}{b} \cdot p,$$
$$z \geqslant \frac{b}{c} \cdot p + \frac{a}{c} \cdot q.$$

Adding up above three inequalities, we get

$$x + y + z \geqslant p\left(\frac{c}{b} + \frac{b}{c}\right) + q\left(\frac{c}{a} + \frac{a}{c}\right) + r\left(\frac{b}{a} + \frac{a}{b}\right)$$
$$\geqslant 2(p + q + r).$$

□

The following proof has been noticed much early. The key to the proof is to consider the bisectors in the triangle and applying the embedding inequality.

Proof. 4. (See Figure 5.4.) Denote $\angle BPC = 2\alpha$, $\angle CPA = 2\beta$, $\angle APB = 2\gamma$. Let their bisectors be ω_a, ω_b, ω_c respectively. We only need to prove the following stronger inequality

$$x + y + z \geqslant 2(\omega_a + \omega_b + \omega_c).$$

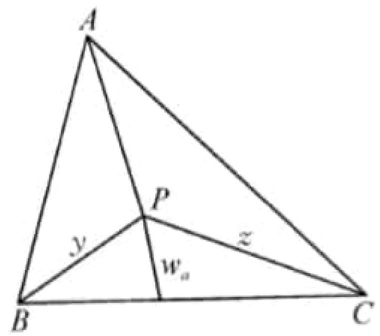

Figure 5.4

By the formula of angle bisector, we obtain

$$\omega_a = \frac{2yz}{y+z}\cos\frac{1}{2}\angle BPC \leqslant \sqrt{yz}\cos\alpha.$$

Similarly

$$\omega_b \leqslant \sqrt{zx}\cos\beta,$$

$$\omega_c \leqslant \sqrt{xy}\cos\gamma.$$

Since $\alpha+\beta+\gamma=\pi$, by the embedding inequality, we conclude that

$$\begin{aligned}2(\omega_a+\omega_b+\omega_c) &\leqslant 2(\sqrt{yz}\cos\alpha+\sqrt{zx}\cos\beta+\sqrt{xy}\cos\gamma)\\ &\leqslant x+y+z.\end{aligned}$$

□

Kang Jiayin, told me the following proof when he was in Grade 2 of Shenzhen High School. He was elected for National Team in 2003.

Proof. 5. (See Figure 5.5.) Make $DT_1 \perp FP$, $ET_2 \perp FP$, the feet are T_1, T_2, respectively.

Since

$$DE \geqslant DT_1+ET_2,\ DT_1=p\sin B,$$
$$ET_2=q\sin A,$$

we have

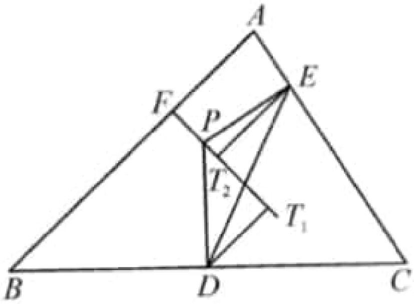

Figure 5.5

$$\begin{aligned}z=\frac{DE}{\sin C} &\geqslant \frac{p\sin B+q\sin A}{\sin C}\\ &= p\frac{\sin B}{\sin C}+q\frac{\sin A}{\sin C}.\end{aligned}$$

Then

$$\begin{aligned}&x+y+z\\ &=PA+PB+PC\\ &\geqslant \left(p\frac{\sin B}{\sin C}+q\frac{\sin A}{\sin C}\right)+q\left(\frac{\sin C}{\sin A}+r\frac{\sin B}{\sin A}\right)+\left(r\frac{\sin A}{\sin B}+q\frac{\sin C}{\sin B}\right)\\ &=p\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}\right)+q\left(\frac{\sin A}{\sin C}+q\frac{\sin C}{\sin A}\right)+r\left(\frac{\sin B}{\sin A}+q\frac{\sin A}{\sin B}\right)\\ &\geqslant 2(p+q+r).\end{aligned}$$

□

Remark. There have been lots of results about the Erdös-Mordell's inequality. Its generalization on plane is easy, which had been finished early by N. Ozeki and H. Vigler. Later it was rediscovered by others

many times. While its generalization in space, especially in n-dimensional space, is difficult. As far as I know, the ideal result has not been obtained.

Example 2. Denote a, b and c the three sides of $\triangle ABC$, then

$$h_a + m_b + t_c \leqslant \frac{\sqrt{3}}{2}(a+b+c), \tag{2}$$

where h_a, m_b and t_c are the altitude of BC, the mid-line of AC and the bisector of $\angle C$ respectively.

Proof. (See Figure 5.6.) Consider the bisector t_a of $\angle A$ instead of altitude h_b. We are to prove a stronger inequality:

$$t_a + m_b + t_c \leqslant \frac{\sqrt{3}}{2}(a+b+c). \tag{a}$$

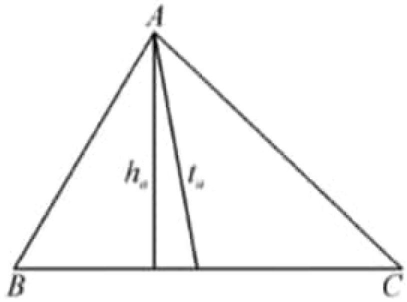

Figure 5.6

In order to prove (a), it suffices to prove a partial inequality

$$m_b + 2t_a \leqslant \frac{\sqrt{3}}{2}(b+2c). \tag{b}$$

If (b) is true, similarly we have

$$m_b + 2t_c \leqslant \frac{\sqrt{3}}{2}(b+2a). \tag{c}$$

Adding up (b) and (c), we obtain (a). So we need only to prove (b). By the formula of angle bisector, we have

$$\begin{aligned} t_a^2 &= \frac{4}{(b+c)^2} \cdot bcp(p-a) \\ &\leqslant p(p-a) = \frac{1}{4}((b+c)^2 - a^2). \end{aligned} \tag{d}$$

Notice that

$$m_b^2 = \frac{1}{4}(2a^2 + 2c^2 - b^2). \tag{e}$$

By Cauchy's inequality and (d), (e), we conclude that

$$\begin{aligned} m_b + 2t_a &\leqslant \sqrt{3(m_b^2 + 2t_a^2)} \\ &\leqslant \sqrt{\frac{3}{4}(2a^2 + 2c^2 - b^2 + 2(b + c)^2 - 2a^2)} \\ &= \frac{\sqrt{3}}{2}(b + 2c), \end{aligned}$$

which is (b). □

Remark. (1) Carefully observing various proofs of Examples 1 and 2, we assure that it is a common technique to consider a part of linear geometric inequality. The aim of various proofs of Example 1 is to obtain the part of inequality

$$x \geqslant \lambda_1 q + \lambda_2 r.$$

λ_1, λ_2 are nothing to do with the moving point P. While in Example 2 we obtain the result by find the local inequality

$$m_b + 2t_a \leqslant \frac{\sqrt{3}}{2}(b + 2c).$$

(2) In Example 2, we make stronger proposition by consider the angle bisector instead of the altitude. This useful technique is adopted in Proof 4 to Example 1, which will be adopted again in Example 5 of the last chapter "Tetrahedral Inequality".

Example 3. Given an acute triangle ABC. Denote h_a, h_b, h_c the altitude of sides BC, CA, AB, respectively, and s the semi-circumference. Then

$$\sqrt{3} \cdot \max\{h_a, h_b, h_c\} \geqslant s.$$

Equality holds if $\triangle ABC$ is equilateral.

Proof. If $\triangle ABC$ is not equilateral, the problem can be changed into a problem for the isosceles triangle.

In fact, if $\angle A \geqslant \angle B > \angle C$, then $\angle A > \frac{\pi}{3}$, and $h_c > h_b \geqslant h_a$. Denote h the longest altitude h_c (see Figure 5.7). Extend the shortest side AB to D with $AD = AC$ and link CD. If $\sqrt{3}h \geqslant s$ holds for isosceles $\triangle ACD$, then it holds for general acute triangle.

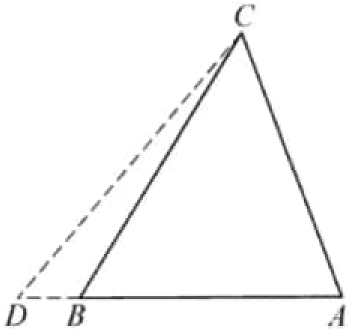

Figure 5.7

We first prove $\sqrt{3}h \geqslant s$ for the isosceles triangle. Since

$$s = AC + \frac{1}{2}CD,\ CD = 2AC \cdot \sin\frac{A}{2},\ h_c = AC \cdot \sin A,$$

we conclude that $\sqrt{3}h \geqslant s$ is equivalent to

$$\sqrt{3}\sin A \geqslant 1 + \sin\frac{A}{2}\ \left(\frac{\pi}{3} < A < \frac{\pi}{2}\right). \tag{a}$$

Denote $x = \sin\frac{A}{2}$, then $\frac{1}{2} < x < \frac{\sqrt{2}}{2}$, (a) changes into

$$12x^4 - 11x^2 + 2x + 1 \leqslant 0.$$

Namely

$$(2x-1)(x+1)(6x^2-3x-1) \leqslant 0. \tag{b}$$

Notice that the range of variable x, it is easy to see $2x - 1 > 0$, $x + 1 > 0$, $6x^2 - 3x - 1 \leqslant 0$, so that (b) follows. □

Remark. The technique of change the general triangle into isosceles triangle is worthy to be noticed. It greatly simplifies the problem.

Example 4 (Zirakzadeh's inequality). Suppose that points P, Q, R lie on three sides BC, CA, AC and trisect the perimeter of $\triangle ABC$, then

$$QR + RP + PQ \geqslant \frac{1}{2}(a + b + c).$$

Proof. We adopt the following projection method to produce a part of linear geometric inequality.

(See Figure 5.8.) Draw two lines from points Q and R perpendicular to line BC, the feet are M and N, respectively, then

$$QR \geqslant MN = a - (BR \cdot \cos B + CQ \cdot \cos C).$$

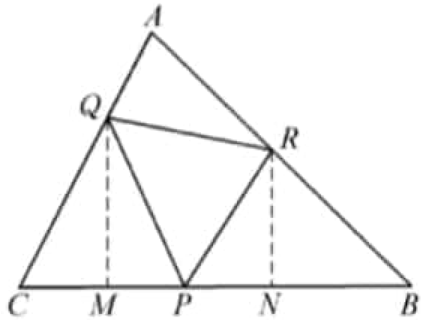

Figure 5.8

Similarly

$$RP \geqslant b - (CP \cdot \cos C + AR \cdot \cos A),$$
$$PQ \geqslant c - (AQ \cdot \cos A + BP \cdot \cos B).$$

Adding up above three inequalities and notice that

$$AQ + AR = BR + BP = CP + CQ = \frac{1}{3}(a + b + c),$$

we have,

$$QR + RP + PQ \geqslant \frac{1}{3}(a + b + c)(3 - \cos A - \cos B - \cos C),$$

by

$$\cos A + \cos B + \cos C \leqslant \frac{3}{2},$$

we conclude that

$$QR + RP + PQ \geqslant \frac{1}{2}(a + b + c).$$

□

Remark. The above beautiful answer was given by Mr. Yang Xuezhi. This problem ever caused extensive discussion.

The following difficult problem of Example 5 was found and proved by Mr. Wang Zhen.

Example 5. Suppose that I, G are the incenter and the barycenter of $\triangle ABC$, respectively, then

$$AI + BI + CI \leqslant AG + BG + CG.$$

Proof. Denote $BC = a$, $AC = b$, $AB = c$. Without loss of generality, we may assume that $a \geqslant b \geqslant c$. (See Figure 5.9.) We will prove that G must lie on $\triangle BIC$.

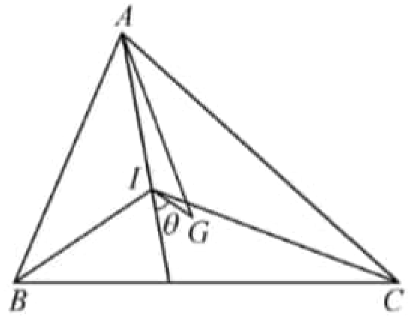

Figure 5.9

Firstly, we will prove that G can not lie in $\triangle AIB$. Otherwise suppose G were in $\triangle AIB$, then

$$S_{\triangle ABG} < S_{\triangle AIB}.$$

Notice that

$$S_{\triangle ABG} = \frac{1}{3}S_{\triangle ABC}, \qquad \frac{S_{\triangle AIB}}{S_{\triangle ABC}} = \frac{c}{a+b+c} \leqslant \frac{1}{3},$$

we obtain

$$S_{\triangle AIB} \leqslant \frac{1}{3}S_{\triangle ABC} = S_{\triangle ABG}.$$

It is a contradiction.

Secondly, we prove G cannot lie in $\triangle AIC$. Otherwise suppose G were in $\triangle AIC$. Suppose that CI intersects AB at T, and CG intersects AB at L, then $AT > AL$.

Notice that

$$AL = BL, \qquad \frac{AT}{BT} = \frac{b}{a} \leqslant 1,$$

therefore

$$AT \leqslant \frac{1}{2}AB = AL.$$

It is a contradiction.

We conclude that G lies on $\triangle BIC$, furthermore G lies on the right side of AI. Let θ be the supplementary angle of $\angle AIG$, then $0 \leqslant \theta \leqslant \frac{A+C}{2}$. We have

$$AG \geqslant AI + GI\cos\theta.$$

Similarly

$$BG \geqslant BI + GI\cos\left(90^\circ + \frac{C}{2} - \theta\right),$$

$$CG \geqslant CI - GI\cos\left(\frac{A+C}{2} - \theta\right).$$

Therefore

$$\begin{aligned}
&AG + BG + CG - (AI + BI + CI) \\
&\geqslant GI\left(\cos\theta + \cos\left(90^\circ + \frac{C}{2} - \theta\right) - \cos\left(\frac{A+C}{2} - \theta\right)\right) \\
&= GI\left(\cos\theta - 2\sin\frac{B+C}{4}\cos\left(\frac{B-C}{4} + \theta\right)\right).
\end{aligned}$$

Notice that

$$\frac{B+C}{4} \leqslant 30^\circ,\ \theta \leqslant \frac{B-C}{4} + \theta < 90^\circ,$$

we have

$$\cos\theta - 2\sin\frac{B+C}{4}\cos\left(\frac{B-C}{4} + \theta\right) \geqslant \cos\theta - \cos\left(\frac{B-C}{4} + \theta\right) \geqslant 0,$$

so that

$$AG + BG + CG \geqslant AI + BI + CI.$$

□

Exercises 5

1. Let G be the barycenter of $\triangle ABC$. AG, BG, CG intersect circumcircle of $\triangle ABC$ at points A_1, B_1, C_1, respectively. Then

$$GA_1 + GB_1 + GC_1 \geqslant GA + GB + GC.$$

Equality holds if and only if $\triangle ABC$ is equilateral.

2. For given four points in a convex quadrangle, show that there

is a point on boundary of the quadrangle, so that the sum of distance from the point to vertexes of quadrangle is greater than that of the distance from it to four given points. (A problem of St. Petersburg Mathematical Contest in 1993.)

3. Suppose that $ABCDEF$ is a convex hexagon, and $AB \parallel ED$, $BC \parallel FE$, $CD \parallel AF$. Denote R_A, R_C, R_E the radius of circumcircle of $\triangle FAB$, $\triangle BCD$, $\triangle DEF$ respectively, and p is the perimeter of hexagon, show that

$$R_A + R_C + R_E \geqslant \frac{p}{2}.$$

(A problem of 37th IMO.)

4. (Cavachi) Suppose a is the longest side of convex hexagon $ABCDEF$, and $d = \min\{AD, BE, CF\}$, then

$$d \leqslant 2a.$$

5. (Zhu Jiegen) Suppose that I is the incencer of $\triangle ABC$. Denote r_1, r_2, r_3 the radius of inscribed circle of $\triangle IBC$, $\triangle ICA$, $\triangle IAB$, respectively, show that

$$3\sqrt{3}(2-\sqrt{3})r \leqslant r_1 + r_2 + r_3 \leqslant \frac{3\sqrt{3}(2-\sqrt{3})}{2}R,$$

where r, R are the radius of inscribed circle and circumcircle of $\triangle ABC$, respectively.

Chapter 6 Algebraic methods

So far the methods we used are mostly of geometric and triangular. In this section we mainly introduce the algebraic method.

It is convenient to construct algebra identities to prove some distance inequalities. The following typical inequality was given by M.S. Klamkin at his early times.

Example 1. On a plane, there is a $\triangle ABC$ and a point P. Show that

$$a \cdot PB \cdot PC + b \cdot PC \cdot PA + c \cdot PA \cdot PB \geqslant abc.$$

Proof. We consider the plane as a complex plane. Let P, A, B, C correspond to complex numbers z, z_1, z_2, z_3 respectively. Define

$$f(z) = \frac{(z - z_2)(z - z_3)}{(z_1 - z_2)(z_1 - z_3)} + \frac{(z - z_3)(z - z_1)}{(z_2 - z_3)(z_2 - z_1)} + \frac{(z - z_1)(z - z_2)}{(z_3 - z_1)(z_3 - z_2)},$$

then $f(z)$ is a quadratic polynomial of z. Notice that

$$f(z_1) = f(z_2) = f(z_3) = 1,$$

hence $f(z) \equiv 1$. So that

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab}$$
$$= \left|\frac{(z - z_2)(z - z_3)}{(z_1 - z_2)(z_1 - z_3)}\right| + \left|\frac{(z - z_3)(z - z_1)}{(z_2 - z_3)(z_2 - z_1)}\right| + \left|\frac{(z - z_1)(z - z_2)}{(z_3 - z_1)(z_3 - z_2)}\right|$$
$$\geqslant |f(z)| = 1.$$

□

The following interesting problem was proposed by Tweedie.

Example 2. Suppose $\triangle ABC$, $\triangle A'B'C'$ are equilateral triangles on a plane with the same direction of vertex array, then the sum of any two of line segments AA', BB', CC' is greater than or equal to the third one.

Proof. (See Figure 6.1.) Since $\triangle ABC$, $\triangle A'B'C'$ are similar and with the same direction of vertex array, then

$$(z_1'-z_1)(z_2-z_3)+(z_2'-z_2)(z_3-z_1)+(z_3'-z_3)(z_1-z_2)=0,$$

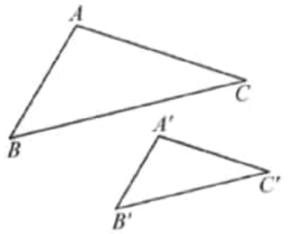

Figure 6.1

where A, B, C correspond to complex numbers z_1, z_2, z_3; A', B', C' correspond to complex numbers z_1', z_2', z_3'. By the property of complex norm, we get

$$|z_1'-z_1|\cdot|z_2-z_3|+|z_2'-z_2|\cdot|z_3-z_1|\geqslant|(z_3'-z_3)(z_1-z_2)|.$$

Notice that $\triangle ABC$ is equilateral, so that

$$|z_2-z_3|=|z_3-z_1|=|z_1-z_2|.$$

Therefore

$$|z_1'-z_1|+|z_2'-z_2|\geqslant|z_3'-z_3|.$$

Namely

$$AA'+BB'\geqslant CC'.$$

Similarly we can get the other two inequalities. □

Now we recall a simple proposition in plane geometry: three positive numbers a, b, c can be three sides of a triangle if and only if there exists three positive numbers x, y, z such that $a=y+z$, $b=x+z$, $c=x+y$. (The sufficiency of this conclusion can be verified directly. Decomposition of

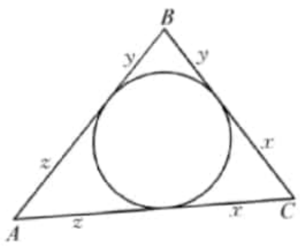

Figure 6.2

Figure 6.2 shows the necessity.)

By this conclusion, we can consider the inequality of x, y, z instead of inequality of triangle sides a, b, c by the identities $x=-a+b+c$, $y=a-b-c$, $z=a+b-c$.

The solution of following question in the 24th IMO is a typical application of the above method.

In $\triangle ABC$, show that $b^2c(b-c)+c^2a(c-a)+a^2b(a-b)\geqslant 0$.

A concise proof using the above method is to use the inequality:

$$\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)(y+z+x)\geqslant(x+y+z)^2,$$

y, z such that $a = y + z$, $b = x + z$, $c = x + y$. (The sufficiency of this conclusion can be verified directly. Decomposition of

Figure 6. 2

Figure 6. 2 shows the necessity.)

By this conclusion, we can consider the inequality of x, y, z instead of inequality of triangle sides a, b, c by the identities $x = -a + b + c$, $y = a - b - c$, $z = a + b - c$.

The solution of following question in the 24th IMO is a typical application of the above method.

In $\triangle ABC$, show that $b^2c(b-c) + c^2a(c-a) + a^2b(a-b) \geqslant 0$.

A concise proof using the above method is to use the inequality:

$$\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right)(y + z + x) \geqslant (x + y + z)^2,$$

by Cauchy inequality. The detail answer can be seen in any IMO tutorial book.

The following problem is a bit new.

Example 3. Denote r_a, r_b, r_c the radius of escribed circles corresponding to three sides a, b, c of $\triangle ABC$ respectively, show that

$$\frac{a^2}{r_b^2 + r_c^2} + \frac{b^2}{r_c^2 + r_a^2} + \frac{c^2}{r_a^2 + r_b^2} \geqslant 2.$$

Proof. By the substitution

$$x = -a + b + c, \quad y = a - b + c, \quad z = a + b - c,$$

so that x, y, $z > 0$. Notice that

$$S_{\triangle ABC} = \frac{1}{4}\sqrt{(x + y + z)xyz},$$

$$r_a = \frac{2S_{\triangle ABC}}{b + c - a} = \frac{1}{2x}\sqrt{(x + y + z)xyz},$$

and so on. The original inequality is equivalent to (after calculating) the following algebraic inequality.

$$\frac{y^2z^2(y+z)^2}{y^2+z^2} + \frac{z^2x^2(z+x)^2}{z^2+x^2} + \frac{x^2y^2(x+y)^2}{x^2+y^2} \geqslant 2xyz(x+y+z). \tag{a}$$

Proof. We establish planar Cartesian coordinate system by taking the line BC for x-axis and the line of the altitude passing A for y-axis (see Figure 6.4).

The coordinates of A, B, C are $(0, a)$, $(-b, 0)$, $(c, 0)$ (where a, b, $c > 0$), then

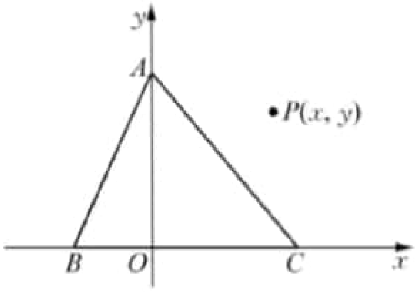

Figure 6.4

$$\tan B = \frac{a}{b}, \qquad \tan C = \frac{a}{c},$$

$$\tan A = -\tan(B + C) = \frac{a(b+c)}{a^2 - bc}.$$

By $\angle A$ is acute angle, we have $a^2 - bc > 0$.

Suppose the coordinate of P is (x, y), then

$$\begin{aligned}
& u^2 \tan A + v^2 \tan B + w^2 \tan C \\
= {} & [x^2 + (y-a)^2]\frac{a(b+c)}{a^2 - bc} + \frac{a}{b}[(x+b)^2 + y^2] + \frac{a}{c}[(x-c)^2 + y^2] \\
= {} & (x^2 + y^2 + a^2 - 2ay)\frac{a(b+c)}{a^2 - bc} + \frac{a(b+c)}{bc}(x^2 + y^2 + bc) \\
= {} & \frac{a(b+c)}{bc(a^2 - bc)}[a^2x^2 + (ay - bc)^2 + 2bc(a^2 - bc)] \\
\geqslant {} & \frac{a(b+c)}{bc(a^2 - bc)} \cdot 2bc(a^2 - bc) \\
= {} & 2a(b+c) = 4\Delta.
\end{aligned}$$

From above proof, we see that equality holds if and only if $x = 0$ and $y = bc/a$, namely, P is the orthocentre $(0, bc/a)$ of $\triangle ABC$. □

The discovery and construction of algebraic identities is the most basic method to find and prove geometric inequalities.

Example 6. Let a, b, c be three sides of $\triangle ABC$, respectively. a', b', c' are three sides of $\triangle A'B'C'$. $S_{\triangle ABC} = F$ and $S_{\triangle A'B'C'} = F'$. Suppose that

Let (b), (c) be multiplied by θ and $1-\theta$, respectively, then adding them up together, we have

$$H \geqslant 8\left[\lambda F'^2 + \left(\frac{\theta}{\mu} + \frac{1-\theta}{\nu}\right)F^2\right]. \tag{d}$$

It is easy to see

$$\frac{\theta}{\mu} + \frac{1-\theta}{\nu} \geqslant \frac{1}{\lambda}.$$

Then by (d), it follows that

$$H \geqslant 8\left(\lambda F'^2 + \frac{1}{\lambda}F^2\right).$$

□

Remark. It is the key difficulty to find algebraic identity (a) in above example. This example given by Mr. Chen Ji is a strengthened form to Neuberg-Pedoe's inequality.

Exercises 6

1. Let p be any point in acute $\triangle ABC$, then

$$PA \cdot PB \cdot AB + PB \cdot PC \cdot BC + PC \cdot PA \cdot CA \geqslant AB \cdot BC \cdot CA.$$

Equality holds if and only if P is the orthocenter of $\triangle ABC$.

2. To make two squares $ABDE$ and $ACFG$ outward with sides AB and CD of $\triangle ABC$, respectively. $BP \perp BC$, $CQ \perp BC$, the feet are P, Q, then

$$BP + CQ \geqslant BC + EG.$$

Equality holds if and only if $AB = AC$.

3. Let $P(z)$ be correspond to complex number z on the complex plane. Complex number $a = p + \mathrm{i}q$ (p, $q \in \mathbf{R}$). $P(z_1)$, ..., $P(z_5)$ are the vertexes of convex pentagon Q. Furthermore, the origin and $P(az_1)$, ..., $P(az_5)$ lie in the interior of Q. Prove that

$$p+q\cdot\tan\frac{\pi}{5}\leqslant 1.$$

4. Suppose that a, b, c are three sides of $\triangle ABC$, and h_a, h_b, h_c are altitudes corresponding to a, b, c, respectively, and r_a, r_b, r_c are radius of escribed circle corresponding to a, b, c, respectively. Prove that

$$\left(\frac{h_a}{r_b}\right)^2+\left(\frac{h_b}{r_c}\right)^2+\left(\frac{h_c}{r_a}\right)^2\geqslant 4\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right).$$

Equality holds if and only if $\triangle ABC$ is equilateral.

5. (Wen Jiajin) Suppose that AD, BE, CF are the angle bisectors of $\triangle ABC$. The square roots of the distances from the moving point P within the $\triangle ABC$ to the three sides are the lengths of sides of a triangle. Prove that.

(1) The orbit of P is in the interior of a ellipse Γ. And Γ is tangent to the three sides BC, AB, AC of $\triangle ABC$ at points D, E, F.

(2) The area S_Γ of Γ satisfies

$$\frac{4\sqrt{3}\pi}{9}S_{\triangle DEF}\geqslant S_\Gamma\geqslant\frac{\sqrt{3}\pi}{9}S_{\triangle ABC}.$$

Chapter 7 Isoperimetric and extremal value problem

All kinds of isoperimetric problems in any space seem to be one of the permanent subjects in geometry studying. Isoperimetric theorem indicates that several special geometric graph in plane geometry, such as circle, regular n-gon. That special properties of them are very spectacular. There are several isoperimetric theorem which are usually used in high school mathematic contents.

Theorem 1 (Isoperimetric Theorem 1). Of all plane figures with given circumference, the circle has the largest area. And of all plane figures with given area, the circle has the least circumference.

Theorem 2 (Isoperimetric Theorem 2). Of all plane n-gons with given circumference, the regular n-gon has the largest area. And of all plane n-gons with given area, the regular n-gon has the least circumference.

Let $a_1, a_2, \ldots, a_n$ be the side length of a n-gon with area F, then the following isoperimetric inequality is inferred from Isoperimetric Theorem

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant 4n\tan\frac{\pi}{n} \cdot F.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

The following Theorem 3 is a generalization of extremal value properties of inscribed quadrilateral in a circle in Chapter 3.

Theorem 3 (Steiner's Theorem). For all of n-gons with given sides, the one with a circumcircle has the largest area.

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant 4n\tan\frac{\pi}{n}\cdot F.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

The following Theorem 3 is a generalization of extremal value properties of inscribed quadrilateral in a circle in Chapter 3.

Theorem 3 (Steiner's Theorem). For all of n-gons with given sides, the one with a circumcircle has the largest area.

Theorem 4. Of all inscribed n-polygon in a circle, the regular n-gon has the largest area.

Let a_1, a_2, ..., a_n be lengths of sides of inscribed n-gon in a circle which radius R, then by Theorem 4, we can infer that

$$\sum_{i=1}^{n} a_i \leqslant 2nR\cdot\sin\frac{\pi}{n},$$

equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Firstly, we discuss several simple examples.

Example 1. Let r be the radius of a circle. l be the tangent line of the circle through a given point P. Through a moving point R in the circle, make a line RQ perpendicular to l with intersection point Q. Try to find the maximum area of $\triangle PQR$. (The 13th Canada's mathematic problem.)

Answer. (See Figure 7.1.)

Notice that $OP \parallel RQ$, make a line $RS \parallel l$, S is the intersection point with $\odot O$. Join with PS, it is easy to prove that

$$S_{\triangle PQR} = \frac{1}{2}S_{\triangle PRS}.$$

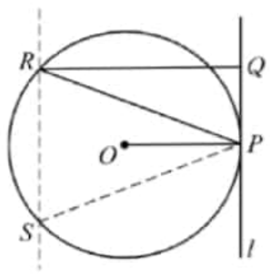

Figure 7.1

By Theorem 4, if inscribed triangle $\triangle PRS$ in a circle is regular, then $\max\{S_{\triangle PRS}\} = \frac{3\sqrt{3}}{4}r^2$, so that $S_{\triangle PQR} = \frac{3\sqrt{3}r^2}{8}$.

The next Example is a familiar problem.

Proof. Reflect Q on the longest side to form a hexagon (see Figure 7.4).

(In some special cases will form pentagon or rectangle, but the proof is similar.) Since the circumference of this convex hexagon is a given value $2(a+b+c)$, the area $2F$ of the hexagon satisfy

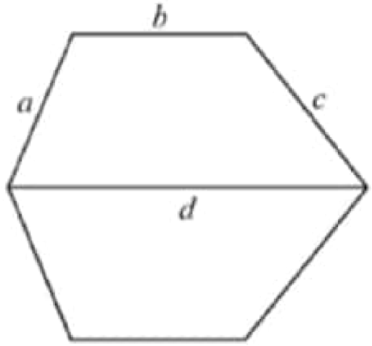

Figure 7.4

$$2F \leqslant \frac{3\sqrt{3}}{2}\left(\frac{a+b+c}{3}\right)^2,$$

by Isoperimetric Theorem 2.

Applying a, $b \leqslant c$, we obtain

$$F \leqslant \frac{3\sqrt{3}}{4}c^2.$$

□

Remark. From the above method, we can generalize Popa's inequality to n-gon. (Refer to Problem 5 under Exercises in this chapter.)

By the above method, Ms. Zhang Haijuan and the author construct an inequality about two n-gons as follows.

Example 4. Let Ω_1, Ω_2 be two n-gons with areas F_1, F_2 and sides $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n-1} \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{n-1} \leqslant b_n$, respectively. Show that

$$\frac{F_1}{a_n^2} + \frac{F_2}{b_n^2} < \frac{(n-1)^2}{4\pi}\left(\frac{a_{n-1}}{a_n} + \frac{b_{n-1}}{b_n}\right)^2.$$

Proof. Let A_nA_1 be the longest side of Ω_1. Taking A_nA_1 as one side, construct a polygon $\Omega'_2 \sim \Omega_1$ and Ω'_2, Ω_1 locates on different side of A_nA_1. The longest side $B'_nB'_1$ of Ω'_2 superposes A_nA_1. (See Figure 7.5.)

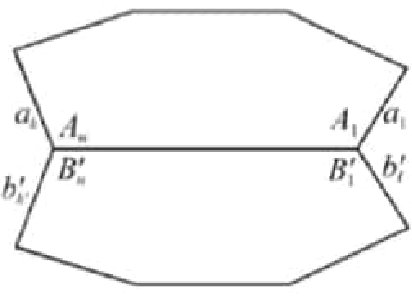

Figure 7.5

Let $\widetilde{\Omega} = \Omega_1 \cup \Omega_2$, then the circumference of $\widetilde{\Omega}$ is

$$\sum_{i=1}^{n-1}\left(a_i + \frac{a_n}{b_n}b_i\right)$$

and the area is $F_1 + \frac{a_n^2}{b_n^2}F_2$.

Let $\angle A_n$, $\angle A_1$ and $\angle B_n$, $\angle B_1$ be two angles of Ω_1, Ω_2, respectively, while the longest side A_nA_1, B_nB_1 belongs to Ω_1, Ω_2, respectively. Let another sides of $\angle A_n$, $\angle A_1$ be a_k, a_1 respectively and also another sides of $\angle B_n$, $\angle B_1$ are $b'_{k'}$, $b'_{l'}$ respectively.

If $\angle A_n$, $\angle A_1$ and $\angle B_n$, $\angle B_1$ are not supplementary angle, respectively, then the sides a_k, a_1 and $b'_{k'}$, $b'_{l'}$ are not in a line, respectively. Then $\widetilde{\Omega}$ is a $2(n-1)$-gon. Applying isoperimetric inequality (Theorem 2), we obtain

$$\begin{aligned}
F_1 + \frac{a_n^2}{b_n^2}F_2 &\leqslant \frac{\left(\sum_{i=1}^{n-1}\left(a_i + \frac{a_n}{b_n}b_i\right)\right)^2}{8(n-1)} \cdot \cot\frac{\pi}{2(n-1)} \\
&\leqslant \frac{\left((n-1)a_{n-1} + (n-1)\frac{a_n}{b_n}b_{n-1}\right)^2}{8(n-1)} \cdot \cot\frac{\pi}{2(n-1)} \\
&= \frac{(n-1)\left(a_{n-1} + \frac{a_n}{b_n}b_{n-1}\right)^2}{8} \cdot \cot\frac{\pi}{2(n-1)},
\end{aligned}$$

namely,

$$\frac{F_1}{a_n^2} + \frac{F_2}{b_n^2} \leqslant \frac{(n-1)}{8}\left(\frac{a_{n-1}}{a_n} + \frac{b_{n-1}}{b_n}\right)^2 \cdot \cot\frac{\pi}{2(n-1)}. \tag{a}$$

If the supplementary angle of $\angle A_n$ or $\angle A_1$ is $\angle B_n$ or $\angle B_1$, then $\widetilde{\Omega}$ is a $(2n-3)$-gon. If the supplementary angle of $\angle A_n$, $\angle A_1$ are $\angle B_n$, $\angle B_1$, then $\widetilde{\Omega}$ is a $2(n-2)$-gon. With the same discussion as above, applying isoperimetric inequality, we obtain the following results

$$\frac{F_1}{a_n^2} + \frac{F_2}{b_n^2} \leqslant \frac{(n-1)^2}{4(2n-3)}\left(\frac{a_{n-1}}{a_n} + \frac{b_{n-1}}{b_n}\right)^2 \cdot \cot\frac{\pi}{2n-3}, \tag{b}$$

Hence

$$\sum_{i=1}^{n} a_i^2 = 2\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i x_{i+1}\cos\left(\varphi_i + \frac{\pi}{n}\right). \tag{a}$$

By the Cauchy inequality and isoperimetric inequality, we have

$$\sum_{i=1}^{n} a_i^2 \geqslant \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant 4F\tan\frac{\pi}{n}. \tag{b}$$

By (a) and (b), we conclude

$$\begin{aligned}
\sum_{i=1}^{n} x_i^2 &\geqslant 2F\tan\frac{\pi}{n} + \sum_{i=1}^{n} x_i x_{i+1}\cos\left(\varphi_i + \frac{\pi}{n}\right) \\
&= \sum_{i=1}^{n}\left[x_i x_{i+1}\sin\left(\varphi_i + \frac{\pi}{n}\right)\tan\frac{\pi}{n} + x_i x_{i+1}\cos\left(\varphi_i + \frac{\pi}{n}\right)\right] \\
&= \sec\frac{\pi}{n}\sum_{i=1}^{n} x_i x_{i+1}\left[\sin\left(\varphi_i + \frac{\pi}{n}\right)\sin\frac{\pi}{n} + \cos\frac{\pi}{n}\cos\left(\varphi_i + \frac{\pi}{n}\right)\right] \\
&= \sec\frac{\pi}{n}\sum_{i=1}^{n} x_i x_{i+1}\cos\varphi_i.
\end{aligned}$$

□

Remark. (1) Ozeki's inequality was proposed by N. Ozeki when he generalized the famous Erdös-Mordell inequality to polygon. The related results about it could refer to N. Ozeki "On the P. Erdös inequality for the triangle", J. College Arts Sci. Chiba Univ, 1957 (2):247-250. This inequality was recovered by Lenhard in 1961.

(2) Ozeki's inequality is a generalization of triangle imbedding inequality to polygon. It can be generalized to 3 and n-dim spaces. The results for tetrahedron are as follows:

Let θ_{ij} $(1 \leqslant i < j \leqslant 4)$ be inside interfacial angle of tetrahedron Ω, then for each real number $x_1, \ldots, x_4$, we have

$$\sum_{i=1}^{4} x_i^2 \geqslant 2\sum_{1\leqslant i<j\leqslant 4}^{4} x_i x_j \cos\theta_{ij}.$$

Other generalizations and related results could refer to the articles of Mr. Zhang Yao and the author (Linear Algebra and its Applications,

1998;278).

(3) Using the above methods in the proof, we can prove the following algebraic inequality:

Suppose that x, y, $z \geqslant 0$, then

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geqslant (xy+yz+zx)^2.$$

The proof is left to readers.

Exercises 7

1. Among all of triangles with given side BC and its opposite angle α. Show that:

(1) The isosceles triangle with base BC has the largest area.

(2) The isosceles triangle with base BC has the largest circumference.

2. Two equilateral triangles inscribe a circle of radius r. Let K be the area of overlap section of two triangles. Show that

$$2K \geqslant r^2\sqrt{3}.$$

3. Of all quadrilaterals which three sides each with length 1 and one angle $30°$, please find a quadrilateral with the largest area.

4. Let P be a inner point of convex n-gon $A_1A_2\cdots A_n$ and the distance from P to each side A_1A_2, A_2A_3, ..., A_nA_1 are d_1, d_2, ..., d_n, respectively. Show that

$$\sum_{i=1}^{n}\frac{a_i}{d_i} \geqslant 2n\tan\frac{\pi}{n},$$

where $a_i = A_iA_{i+1}(A_{n+1} = A_1)$, and give the necessary and sufficient condition for equalities hold.

5. Let a convex n-gon in a plane with area F satisfy $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, show that

$$F < \frac{(n-1)^2}{2\pi}a_{n-1}^2.$$

Chapter 8 Embed inequality and inequality for moment of inertia

Embed triangle inequality (called Embed inequality in short) plays an important role in research of primary geometric inequalities in recent years, which is a source of generating geometry inequalities. Embed triangle inequality can be read as follows.

Theorem 1 (Embedding Triangle Inequality). Suppose that $A + B + C = (2k+1)\pi$, x, y, $z \in \mathbf{R}$, then

$$x^2 + y^2 + z^2 \geqslant 2yz\cos A + 2zx\cos B + 2xy\cos C, \tag{1}$$

equality holds if and only if $x : y : z = \sin A : \sin B : \sin C$.

Proof. The difference of two sides of (1) is $(x - y\cos C - z\cos B)^2 + (y\sin C - z\sin B) \geqslant 0$. □

By the facial meaning, the geometric explanation that inequality (1) called is: if $0 < A, B, C < \pi$, and for arbitrarily real number x,

equality holds if and only if $x : y : z = \sin A : \sin B : \sin C$.

(3) $(x+y+z)^2 \geqslant 4(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C)$, equality holds if and only if $x : y : z = \sin 2A : \sin 2B : \sin 2C$.

Proof. (1) It can be proved by applying double angle formula $\cos A = 1 - 2\sin^2(A/2)$ to (1).

(2) It can be proved by applying double angle formula $\cos A = 2\cos^2(A/2) - 1$ to (1).

(3) It can be proved by applying half angle formula $\sin^2 A = (1 - \cos 2A)/2$ to (1). □

Remark. Applying sine law to (3), we can get: in $\triangle ABC$, λ, μ, $\upsilon \in \mathbf{R}$, we have

$$(\lambda+\mu+\upsilon)^2R^2 \geqslant \mu\upsilon a^2 + \nu\lambda b^2 + \lambda\mu c^2$$
$$\Leftrightarrow (\lambda+\mu+\upsilon)^2(abc)^2 \geqslant 16\Delta^2(\mu\upsilon a^2 + \nu\lambda b^2 + \lambda\mu c^2) \tag{2}$$

in (2) let $\lambda = xa^2$, $\mu = yb^2$, $\upsilon = zc^2$, $x, y, z \in \mathbf{R}$, we have

$$(xa^2 + yb^2 + zc^2)^2 \geqslant 16\Delta^2(xy + yz + zx), \tag{3}$$

equality holds if and only if

$$\lambda : \mu : \upsilon = (b^2 + c^2 - a^2) : (c^2 + a^2 - b^2) : (a^2 + b^2 - c^2).$$

Inequality (3) has wild applications, it often appears in the study of the elementary geometric inequality (for example Mr. Shan Zun's "Geometric Inequality". Shanghai Education Press, 1980).

There are other researchers who also obtained some algebra inequality more general than (3) as follows.

Example 1. Assume that there are at least two positive number among λ_1, λ_2 and λ_3, and satisfy $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 > 0$, x, y, z are arbitrarily real numbers. Show that,

$$(\lambda_1 x + \lambda_2 y + \lambda_3 z)^2$$
$$\geqslant (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)(2xy + 2yz + 2zx - x^2 - y^2 - z^2)$$

equality holds if and only if $x/(\lambda_2+\lambda_3) = y/(\lambda_1+\lambda_3) = z/(\lambda_1+\lambda_2)$.

Answer 1. Applying embedding inequality. Since there are at least

two positive numbers among λ_1, λ_2, λ_3, and $\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3>0$, we obtain $\lambda_2+\lambda_3>0$. Similarly we can get $\lambda_1+\lambda_2>0$, $\lambda_1+\lambda_3>0$.

Suppose that $\lambda_1+\lambda_2=c^2$, $\lambda_2+\lambda_3=a^2$, $\lambda_1+\lambda_3=b^2(a, b, c>0)$, then

$$\lambda_1=\frac{1}{2}(b^2+c^2-a^2),\ \lambda_2=\frac{1}{2}(a^2+c^2-b^2),\ \lambda_3=\frac{1}{2}(a^2+b^2-c^2).$$

By $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1>0$, we can get

$$(a+b+c)(a-b+c)(a+b-c)(-a+b+c)>0.$$

Thus, a, b, c can form three sides of a triangle, say $\triangle ABC$. We can get that the primitive inequality

$$\Leftrightarrow \sum\lambda_1^2x^2+2\sum\lambda_1\lambda_2xy\geqslant\left(\sum\lambda_1\lambda_2\right)(2xy+2yz+2zx-x^2-y^2-z^2)$$

$$\Leftrightarrow \sum x^2(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)\geqslant\sum\lambda_1\lambda_2(2yz+2xz)$$

$$\Leftrightarrow \sum x^2c^2b^2\geqslant\sum\frac{c^4-a^4-b^4+2a^2b^2}{4}(2yz+2xz)$$

$$\Leftrightarrow \sum(xcb)^2\geqslant\sum yza^2(b^2+c^2-a^2). \tag{a}$$

In fact, by embedding inequality, we can get that

$$\begin{aligned}\sum(xbc)^2&\geqslant\sum 2(yca)(zba)\cdot\cos A\\&=\sum 2(yca)(zba)\cdot\frac{b^2+c^2-a^2}{2bc}\\&=\sum yza^2(b^2+c^2-a^2).\end{aligned}$$

This is formula (a). Therefore the primitive inequality holds.

Equality holds if and only if

$$\begin{aligned}&\frac{xbc}{\sin A}=\frac{yac}{\sin B}=\frac{zab}{\sin C}\\ \Leftrightarrow&\frac{x}{\sin^2A}=\frac{y}{\sin^2B}=\frac{z}{\sin^2C}\\ \Leftrightarrow&\frac{x}{\lambda_2+\lambda_3}=\frac{y}{\lambda_1+\lambda_3}=\frac{z}{\lambda_1+\lambda_2}.\end{aligned}$$

□

Answer 2. Discriminant Method. By the same method to the previous, we obtain $\lambda_2+\lambda_3>0$, $\lambda_1+\lambda_2>0$, $\lambda_1+\lambda_3>0$. Now, the primitive inequality equivalent to the statement that

$$(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)x^2-2[\lambda_3(\lambda_1+\lambda_2)y+\lambda_2(\lambda_1+\lambda_3)z]x+ \\ [(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)y^2+(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)z^2-2\lambda_1(\lambda_2+\lambda_3)yz]\geqslant 0.$$

The expression on the left side of the previous formula is a quadratic function with respect to x, whose coefficient of the quadratic term is positive.

$$\Delta\leqslant 0$$

that is

$$\begin{aligned}&\lambda_3^2(\lambda_1+\lambda_2)^2y^2+\lambda_2^2(\lambda_1+\lambda_3)^2z^2+2\lambda_3\lambda_2(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)yz\\ \leqslant\ &(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)[(\lambda_1+\lambda_2)(\lambda_3+\lambda_2)y^2+(\lambda_1+\lambda_3)(\lambda_3+\lambda_2)z^2-\\ &2\lambda_1(\lambda_3+\lambda_2)yz]\\ \Leftrightarrow\ &(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)[(\lambda_1+\lambda_2)y-(\lambda_1+\lambda_3)z]^2\geqslant 0,\end{aligned}$$

which holds by the conditions of the problem. Thus, the primitive inequality follows immediately. And equality holds if and only if

$$(\lambda_1+\lambda_2)y=(\lambda_1\lambda_3)z\Leftrightarrow\frac{y}{\lambda_1+\lambda_3}=\frac{z}{\lambda_1+\lambda_2}.$$

By the symmetry of x, y, z, we can see that equality holds if and only if

$$\frac{x}{\lambda_2+\lambda_3}=\frac{y}{\lambda_1+\lambda_3}=\frac{z}{\lambda_1+\lambda_2}.$$

□

The Answer 1 of Example 1 was provided by student Xiang Zhen. The Answer 2 was by Mr. Huang Zhiyi (the former student of the High School Affiliated to South China Normal University, who won a gold medal in the 45th IMO).

Next, we will introduce applications of the embedding inequality.

$$x^2+y^2+z^2 \geqslant \frac{1}{3}(a^2+b^2+c^2).$$

Proof. First we consider the case that P is in $\triangle ABC$. Draw three lines from the vertexes A, B and C of $\triangle ABC$ perpendicular to the segment PA, PB and PC, respectively, see Figure 8.1. The three lines are crossing pairwise at A', B' and C'. So, $\angle BPC=\pi-A'$, $\angle APB=\pi-C'$, $\angle APC=\pi-B'$. By cosine law, we can get that

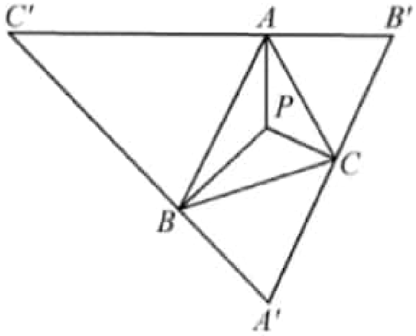

Figure 8.1

$$\begin{aligned} a^2 &= y^2+z^2+2yz\cos A', \\ b^2 &= x^2+z^2+2xz\cos B', \\ c^2 &= x^2+y^2+2xy\cos C'. \end{aligned}$$

Adding up and using the embedding inequality, we obtain

$$\begin{aligned} a^2+b^2+c^2 &= 2(x^2+y^2+z^2)+2xy\cos C'+2xz\cos B'+2yz\cos A' \\ &\leqslant 2(x^2+y^2+z^2)+(x^2+y^2+z^2) \\ &= 3(x^2+y^2+z^2), \end{aligned}$$

as required. For the case that P is on the side of $\triangle ABC$, the proof is similar and is left to the reader. □

We will prove inequality of weighted sine sum, which is a useful inequality set up by Mr. Yang Xuezhi in 1988, by the embedding inequality.

Theorem 3 (Inequality of weighted sine sum). For arbitrary real numbers x, y, z, arbitrary $\triangle ABC$ and positive numbers u, v, w, we have

$$2(yz\sin A+zx\sin B+xy\sin C) \leqslant \left(\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\right)\sqrt{vw+wu+uv}.$$

Equality holds if and only if $x:y:z=\cos A:\cos B:\cos C$ and $u:v:w=\cot A:\cot B:\cot C$.

Proof. Let $x'=\dfrac{x}{\sqrt{u}}$, $y'=\dfrac{y}{\sqrt{v}}$, $z'=\dfrac{z}{\sqrt{w}}$. The primitive

inequality is equivalent to

$$2\sqrt{\frac{vw}{uv+vw+uw}}\cdot y'z'\sin A+2\sqrt{\frac{uw}{uv+vw+uw}}\cdot x'z'\sin B$$
$$+2\sqrt{\frac{uv}{uv+vw+uw}}\cdot x'y'\sin C$$
$$\leqslant x'^2+y'^2+z'^2. \tag{a}$$

By Cauchy inequality, the expression on the left side of the formula (a) is less than or equal to

$$2\sqrt{\frac{vw}{uv+vw+uw}+\frac{uw}{uv+vw+uw}+\frac{uv}{uv+vw+uw}}\cdot$$
$$\sqrt{y'^2z'^2\sin^2A+x'^2z'^2\sin^2B+x'^2y'^2\sin^2C}$$
$$=2\sqrt{y'^2z'^2\sin^2A+x'^2z'^2\sin^2B+x'^2y'^2\sin^2C},$$

thus, it suffices to prove that

$$2\sqrt{y'^2z'^2\sin^2A+x'^2z'^2\sin^2B+x'^2y'^2\sin^2C}\leqslant x'^2+y'^2+z'^2$$
$$\Leftrightarrow 2y'^2z'^2(2\sin^2A-1)+2x'^2z'^2(2\sin^2B-1)+2x'^2y'^2(2\sin^2C-1)$$
$$\leqslant x'^4+y'^4+z'^4$$
$$\Leftrightarrow 2y'^2z'^2\cos(\pi-2A)+2x'^2z'^2\cos(\pi-2B)+2x'^2y'^2\cos(\pi-2C)$$
$$\leqslant x'^4+y'^4+z'^4. \tag{b}$$

Formula (b) can be obtained by the substitutions $(x, y, z)\to(x'^2, y'^2, z'^2)$ and $(A, B, C)\to(\pi-2A, \pi-2B, \pi-2C)$ by the embedding inequality. □

By applying the inequality of weighted sine sum, we can obtain the following two interesting triangle inequalities.

Example 4. Let three sides of $\triangle ABC$ be a, b, c, the corresponding angles be A, B, C. Denotes $s=a+b+c$. And let three sides of $\triangle A'$, B', C' be a', b', c', the corresponding angle be A', B', C'. Denote $s=a+b+c$. Show that

$$\frac{a}{a'}\tan\frac{A}{2}+\frac{b}{b'}\tan\frac{B}{2}+\frac{c}{c'}\tan\frac{C}{2}\geqslant\frac{\sqrt{3}s}{2s'}.$$

Equality holds if and only if $\triangle ABC$ and $\triangle A'B'C'$ are regular triangles.

Proof. Substituting in the inequality of weighted sine sum by $A \to \frac{\pi - A}{2}$, $B \to \frac{\pi - B}{2}$, $C \to \frac{\pi - C}{2}$, we obtain a new inequality, then substituting in the new inequality by $x \to yz$, $y \to zx$, $z \to xy$. And substituting $u \to a'$, $v \to b'$, $w \to c'$, obtain

$$\frac{yz}{a'x} + \frac{xz}{b'y} + \frac{xy}{c'z} \geqslant \frac{2\left(x\cos\frac{A}{2} + y\cos\frac{B}{2} + z\cos\frac{C}{2}\right)}{\sqrt{b'c' + a'c' + a'b'}}. \tag{a}$$

And in (a), we use the substitutions $x \to \frac{bc}{s}\cos\frac{A}{2}$, $y \to \frac{ac}{s}\cos\frac{B}{2}$, $z \to \frac{ab}{s}\cos\frac{C}{2}$, and notice that

$$\frac{\frac{a}{s}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} = \frac{2R\sin A\cos\frac{B}{2}\cos\frac{C}{2}}{2R(\sin A + \sin B + \sin C)\cos\frac{A}{2}}$$

$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}} = \frac{1}{2}\tan\frac{A}{2},$$

and so on. So we can obtain

$$\frac{1}{2}\left(\frac{a}{a'}\tan\frac{A}{2} + \frac{b}{b'}\tan\frac{B}{2} + \frac{c}{c'}\tan\frac{C}{2}\right) \geqslant \frac{2\left(\frac{bc}{s}\cos^2\frac{A}{2} + \frac{ac}{s}\cos^2\frac{B}{2} + \frac{ab}{s}\cos^2\frac{C}{2}\right)}{\sqrt{b'c' + a'c' + a'b'}}. \tag{b}$$

Since

$$b'c' + a'c' + a'b' \leqslant \frac{4}{3}s'^2 \tag{c}$$

and

$$2\left(\frac{bc}{s}\cos^2\frac{A}{2}+\frac{ac}{s}\cos^2\frac{B}{2}+\frac{ab}{s}\cos^2\frac{C}{2}\right)$$

$$=\frac{1}{s}[bc(1+\cos A)+ac(1+\cos B)+ab(1+\cos C)]$$

$$=\frac{1}{2s}[2bc+(a^2+b^2-c^2)+2ac+(a^2+c^2-b^2)+2ab+(-a^2+b^2+c^2)]$$

$$=\frac{1}{2s}[a^2+b^2+c^2+2(ab+bc+ca)]$$

$$=\frac{1}{2s}(a+b+c)^2=\frac{s}{2}. \tag{d}$$

Substitute (c) and (d) into (b), the inequality follows immediately. □

By embedding inequality, we can get well-known Neuberg-Pedoe's inequality and some inequalities about moving point inside the triangle. Limited to the aim of this book, we do not expand the subject in this area any further.

Another well-known inequality with weighted real numbers is the inequality for moment of inertia.

Theorem 4 (Inequality for moment of inertia). Let P be an arbitrary point on the plane of $\triangle ABC$. Denote $PA = R_1$, $PB = R_2$, $PC = R_3$, then for arbitrary real numbers $x, y, z \in \mathbf{R}$, we have

$$(x+y+z)(xR_1^2+yR_2^2+zR_3^2)\geqslant yza^2+xzb^2+xyc^2. \tag{5}$$

Equality holds if and only if $xR_1 : yR_2 : zR_3 = \sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3$, which $\alpha_i = \angle A_{i+1}PA_{i+2}$ ($A_4 = A_1$, $A_5 = A_2$, $i = 1, 2, 3$) (the angles are in the same direction).

Proof. Let $\overrightarrow{PQ}=\dfrac{\sum x\,\overrightarrow{PA}}{\sum x}$, then

$$0\leqslant\left(\sum x\right)^2|\overrightarrow{PQ}|^2=\left|\sum x\,\overrightarrow{PA}\right|^2$$

$$=\sum x^2|\overrightarrow{PA}|^2+2\sum xy\,\overrightarrow{PA}\cdot\overrightarrow{PB}$$